KHOVANOV HOMOLOGY FOR ALTERNATING TANGLES

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Abstract. We describe a "concentration ~~in~~ on the diagonal" condition on the Khovanov complex of tangles, show that this condition is satisfied by the Khovanov complex of the single crossing tangles (⤫) and (⤬), and prove that it is preserved by alternating planar algebra compositions. Hence, this condition is satisfied by the Khovanov complex of all alternating tangles. Finally, in the case of 0-tangles, ~~that is~~ meaning links, our condition is equivalent to a well known result [Lee] which states that the Khovanov homology of a non-split alternating link is supported in two lines.

Thus our condition is a generalization of Lee's theorem to the case of tangles.

Contents

1. Introduction

Khovanov [Kh] constructed an invariant of links which opened new prospects in knot theory and which is now known as the Khovanov homology. Bar-Natan in [BN1] shows how to compute this invariant and found that it is a stronger invariant than the Jones polynomial. Khovanov, Bar-Natan and Garoufalidis [Ga] formulated several conjectures related to the Khovanov homology. One of these refers to the fact that the Khovanov homology of a non-split alternating link is supported in two lines. To see this, in Table 1, we present the dimension of the groups in the Khovanov homology for the Borromean link and illustrate that the no-zero dimension groups are located in two consecutive diagonals. The fact that every alternating link satisfies was proved by Lee in [Lee1].

In [BN2] Bar-Natan presented a new way of seeing the Khovanov homology. In his

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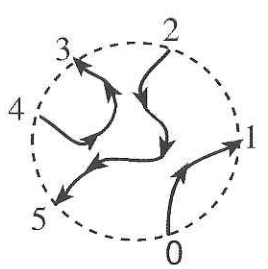
j\i	-3	-2	-1	0	1	2	3
7							1
5						2	
3						1	
1				4	2		
-1			2	4			
-3		1					
-5		2					
-7	1						

Table 1. The Khovanov homology for the Borromean link

approach, a formal chain complex is assigned to every tangle. This formal chain complex, regarded within a special category, is an (up to homotopy) invariant of the tangle. For the particular case in which the tangle is a link, this chain complex coincides with the cube of smoothings presented in [Kh].

This local Khovanov theory was used in [BN3] to make an algorithm which provides a faster computation of the Khovanov homology of a link. The technique used in that last paper was also important for theoretical reasons. We can apply it to prove the invariance of the Khovanov homology, see [BN3]. It was also used in [BN-Mor] to give a simple proof of Lee's result stated in [Lee2], about the dimension of the Lee variant of the Khovanov homology. Here, we will show how it can be used to state a generalization to tangles of the fore-mentioned Lee's theorem [Lee1] about the Khovanov homology of alternating links.

Given an integer $k \geq 0$, a not empty intersection of a link with the 3-ball B having $2k$ boundary points is called a *k-tangle*. Thus, a *k-tangle diagram* consists of k open arcs (strands) and a finite number of loops in the disc D^2. The $2k$ end points of the arcs being $2k$ different fixed points in the boundary of D^2, and with the over/under crossing information represented as usual. A tangle T is *non-split* if every planar isotopy of its tangle diagram produces a connected tangle diagram. A tangle diagram S, having no crossing points is called a *smoothing*. do we need this?



In section 6.1 we observe that the Khovanov complex of an alternating tangle can be endowed with consistent "orientations",1) namely, every strand in every smoothing appearing in the complex can be oriented in a natural way, and likewise every cobordism, in a manner so that these orientations are consistent. (A quick glance on Figures 7 and 8 should suffice to convince the experts). Given an oriented smoothing σ, a point in the boundary of σ can be considered as an *in-boundary* point or an *out-boundary* point depending on the orientation of the respective strand in this point. We can enumerate the boundary points of the σ from 0 to $2k-1$ starting from an in-boundary point of a strand, counting counterclockwise, and finishing in the boundary point to the left of the mentioned in-boundary point. If h_α and t_α denote respectively the numbers assigned to the in-boundary and out-boundary points of an open strand α in an oriented k-strand smoothing σ. Then the *rotation number* of α is given by: $R(\alpha) = \frac{1}{2k}[t_\alpha - h_\alpha]_{2k} - \frac{1}{2}$, where the bracket $[\ ]_{2k}$ is

1) Notice that these are orientations of the {in smoothings}, and they have nothing to do with the orientations of the components of the tangle itself.

defined by

$$[j]_{2k} = \begin{cases} j & \text{if } j > 0 \\ j + 2k & \text{if } j < 0. \end{cases}$$

If α is a loop, $R(\alpha) = 1$ if α is oriented counterclockwise, and $R(\alpha) = -1$ if α is oriented clockwise. The rotation number of σ is the sum of the rotation numbers of its components (loops and strands).

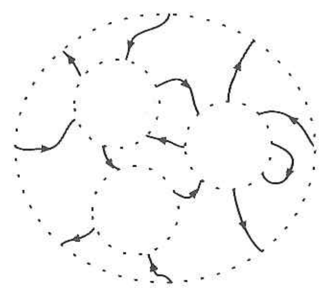
Following [BN2], we define a certain graded category $\mathcal{C}ob^3_o$ of *oriented cobordisms*. The objects of $\mathcal{C}ob^3_o$ are *oriented smoothing*, and the morphisms are oriented cobordisms. This category is used to define the category $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o))$ (abbreviated Kob_o) of complexes over $\mathrm{Mat}(\mathcal{C}ob^3_o)$.

Specifically, for degree-shifted smoothings $\sigma\{q\}$ we define $R(\sigma\{q\}) := R(\sigma) + q$. We further use this *degree-shifted rotation number* to define a special class of chain complexes in Kob_o, of the form

$$\Omega : \qquad \cdots \longrightarrow \left[\sigma^r_j\right]_j \longrightarrow \left[\sigma^{r+1}_j\right]_j \longrightarrow \cdots,$$

which satisfies that for all degree-shifted smoothings $\sigma^r_j\{q\}$, $2r - R(\sigma^r_j\{q\})$ is a constant that we call *rotation constant* of the complex. In other words, twice the homological degrees and the degree-shifted rotation numbers of the smoothings always lie along a single diagonal. Chain complexes satisfying this property are called *diagonal complexes*.

This belongs to the proof, not the statement, of Thm 1.



An important tool for proving Theorem 1 is the concept of *alternating planar algebra*. An alternating planar algebra is an oriented planar algebra as in [BN2, Section 5], where the d-input planar arc diagrams D satisfy the following conditions: i) The number k of strings ending on the external boundary of D is greater than 0. ii) There is complete connection among input discs of the diagram and its arcs, namely, the union of the diagram arcs and the boundary of the internal holes is a connected set. iii) The in- and out-strings alternate in every boundary component of the diagram. A planar arc diagram as this is called a type-$\mathcal{A}$ planar diagram. If Φ is an element in the planar algebra and D is a 1-input type-$\mathcal{A}$ planar diagram then $D(\Phi)$ is called a partial closure of Φ.

Using the above terminology, our main result is stated as follow:

Theorem 1. *If T is an alternating non-split tangle then its Khovanov homology $Kh(T)$ is diagonal and furthermore the same is true for every partial closure of $Kh(T)$.* ?

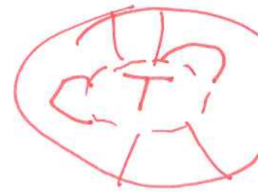
where partial closures mean..

We are say that a complex Ω is *coherently diagonal* if it is a diagonal complex whose partial closures ~~is~~ also diagonal. Indeed, Theorem 1 can be restated as saying that the Khovanov homology $Kh(T)$ of a non-split alternating tangle is coherently diagonal. To prove this theorem we use the fact that non-split alternating tangles form an alternative planar algebra generated for the one-crossing tangles ($\overcrossing$) and ($\undercrossing$). Thus Theorem 1 follows from the observation that $Kh(\overcrossing)$ and $Kh(\undercrossing)$ are coherently diagonal and from Theorem 2 below:

Theorem 2. *If $\Omega_1, \ldots, \Omega_n$ are coherently diagonal complexes and D is an alternating planar diagram then $D(\Omega_1, \ldots, \Omega_n)$ is coherently diagonal*

In the case of alternating tangles with no boundary, i.e., in the case of alternating links, Theorem 1 reduces to Lee's theorem on the Khovanov homology of alternating links.

The work is organized as follows. In section 2, we review Bar-Natan's local Khovanov



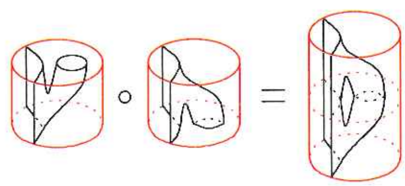
theory. Section 3 is devoted to introduce the category $\mathcal{C}ob^3_o$ an give a quick review of some concepts related to alternating planar algebras. In particular we review the concepts of rotation number, alternating planar diagram, associated rotation number, and basic operators.

Section 4 introduces the concepts of diagonal complexes, coherently diagonal complexes, and their partial closures. We state here some results about the complexes obtained when a basic operator is applied to alternating elements, leading to the proof in section 5 of Theorem 2. Finally section 6 is dedicated to the study of non-split alternating tangles. Here, we prove Theorem 1 and derive from it the Lee theorem formulated in [Lee1].

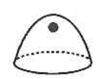
2. The local Khovanov theory: Notation and some details

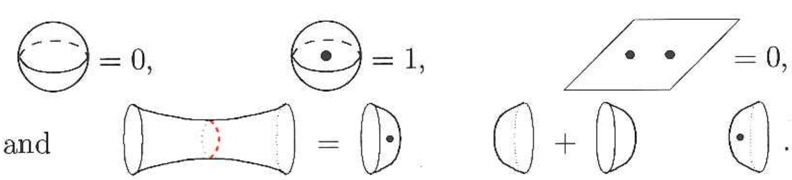
The notation and some results appearing here are treated in more details in [BN2, BN3, Naot]. Given a set B of $2k$ marked points on a circle C, a smoothing with boundary B is a union of strings $a_1, ..., a_n$ embedded in the planar disk for which C is the boundary, such that $\cup_{i=1}^n \partial a_i = B$. These strings are either closed curves, *loops*, or strings whose boundaries are points on B, *strands*. If $B = \emptyset$, the smoothing is a union of circles.

We denote $\mathcal{C}ob^3(B)$, the category whose objects are smoothings with boundary B, and whose morphisms are cobordisms between such smoothings, regarded up to boundary preserving isotopy. The composition of morphisms is given by placing one cobordism atop the other, as shown on the right.



on side

Our ground ring is one in which 2^{-1} exists. The dotted figure  is used as an abbreviation of $\frac{1}{2}$  and $\mathcal{C}ob^3_{\bullet/l}(B)$ represents the category with the same objects and morphisms as $\mathcal{C}ob^3(B)$, whose morphisms are mod out by the local relations:

(1) and 

We will use the notation $\mathcal{C}ob^3$ and $\mathcal{C}ob^3_{\bullet/l}$ as a generic reference, namely, $\mathcal{C}ob^3 = \bigcup_B \mathcal{C}ob^3(B)$ and $\mathcal{C}ob^3_{\bullet/l} = \bigcup_B \mathcal{C}ob^3_{\bullet/l}(B)$. If B has $2k$ elements, we usually write $\mathcal{C}ob^3_{\bullet/l}(k)$ instead of $\mathcal{C}ob^3_{\bullet/l}(B)$. If $\mathcal{C}$ is any category, $\mathrm{Mat}(\mathcal{C})$ will be the additive category whose objects are column vectors (formal direct sums) whose entries are objects of $\mathcal{C}$. Given two objects in this category,

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \\ \vdots \\ \mathcal{O}_n \end{pmatrix} \qquad \mathcal{O}^1 = \begin{pmatrix} \mathcal{O}_1^1 \\ \mathcal{O}_2^1 \\ \vdots \\ \mathcal{O}_m^1 \end{pmatrix},$$

the morphisms between these objects will be matrices whose entries are formal sums of morphisms between them. The morphisms in this additive category are added using the usual matrix addition and the morphism composition is modelled by matrix multiplication, i.e, given two appropriate morphisms $F = (F_{ik})$ and $G = (G_{kj})$ between objects of this category, then $F \circ G$ is given by

$$F \circ G = \sum_k F_{ik} G_{kj}\,,$$

$\mathrm{Kom}(\mathcal{C})$ will be the category of formal complexes over an additive category $\mathcal{C}$. $\mathrm{Kom}_{/h}(\mathcal{C})$ is $\mathrm{Kom}(\mathcal{C})$ modulo homotopy. We also use the abbreviations $\mathrm{Kob}(k)$ and $\mathrm{Kob}_{/h}(k)$ for denoting $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$ and $\mathrm{Kom}_{/h}(\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l}(k)))$.

Objects and morphisms of the categories $\mathcal{C}ob^3$, $\mathcal{C}ob^3_{\bullet/l}$, $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$, $\mathrm{Kob}(k)$, and $\mathrm{Kob}_{/h}(k)$ can be seen as examples of planar algebras, i.e., if D is a n-input planar diagram, it defines an operation among elements of the previously mentioned collections. See [BN2] for specifics of how D defines operations in each of these collections. In particular, if $(\Omega_i, d_i) \in \mathrm{Kob}(k_i)$ are complexes, the complex $(\Omega, d) = D(\Omega_1, \dots, \Omega_n)$ is defined by

$$\begin{aligned} \Omega^r &:= \bigoplus_{r = r_1 + \cdots + r_n} D(\Omega_1^{r_1}, \dots, \Omega_n^{r_n}) \\ d|_{D(\Omega_1^{r_1}, \dots, \Omega_n^{r_n})} &:= \sum_{i=1}^{n} (-1)^{\sum_{j<i} r_j} D(I_{\Omega_1^{r_1}}, \dots, d_i, \dots, I_{\Omega_n^{r_n}}), \end{aligned} \tag{2}$$

$D(\Omega_1, \dots, \Omega_n)$ is used here as an abbreviation of $D((\Omega_1, d_1), \dots, (\Omega_n, d_n))$.

In [BN2] the following very desirable property is also proven. The Khovanov homology is a planar algebra morphism between the planar algebras $\mathcal{T}(s)$ of oriented tangles and $\mathrm{Kob}_{/h}(k)$. That is to say, for an n-input planar diagram D, and suitable tangles $T_1, \dots, T_n$, we have

$$Kh(D(T_1, \dots, T_n)) = D(Kh(T_1), \dots, Kh(T_n)). \tag{3}$$

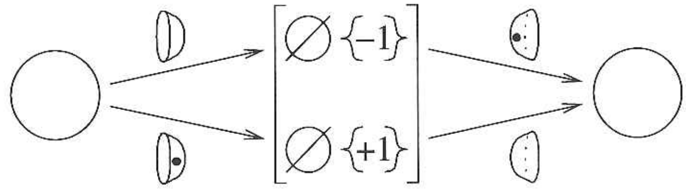
This last property is used in [BN3] to show a local algorithm for computing the Khovanov homology of a link. In that paper, Bar-Natan explained how it is possible to remove the loops in the smoothings, and some terms in the Khovanov complex $Kh(T_i)$ associated to the local tangles $T_1, \dots, T_n$, and then combine them together in an n-input planar diagram D obtaining $D(Kh(T_1), \dots, Kh(T_n))$, and the Khovanov homology of the original tangle.

The elimination of loops and terms can be done thanks to the following: Lemma 4.1 and Lemma 4.2 in [BN3]. We copy these lemmas verbatim:

Lemma 2.1. *(Delooping) If an object S in $\mathcal{C}ob^3_{\bullet/l}$ contains a closed loop ℓ, then it is isomorphic (in $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/l})$) to the direct sum of two copies $S'\{+1\}$ and $S'\{-1\}$ of S in which ℓ is removed, one taken with a degree shift of $+1$ and one with a degree shift of -1. Symbolically, this reads $\bigcirc \equiv \emptyset\{+1\} \oplus \emptyset\{-1\}$.*

The isomorphisms for the proof can be seen in:

(4)



using all the relations in (1).

Lemma 2.2. *(Gaussian elimination, made abstract) If $\phi : b_1 \to b_2$ is an isomorphism (in some additive category $\mathcal{C}$), then the four term complex segment in* $\mathrm{Mat}(\mathcal{C})$

$$(5) \qquad \cdots\ [C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta \\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]\ \cdots$$

is isomorphic to the (direct sum) complex segment

$$(6) \qquad \cdots\ [C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]\ \cdots .$$

Both these complexes are homotopy equivalent to the (simpler) complex segment

$$(7) \qquad \cdots\ [C] \xrightarrow{(\beta)} [D] \xrightarrow{\left(\epsilon - \gamma\phi^{-1}\delta\right)} [E] \xrightarrow{(\nu)} [F]\ \cdots .$$

Here C, D, E and F are arbitrary columns of objects in $\mathcal{C}$ and all Greek letters (other than ϕ) represent arbitrary matrices of morphisms in $\mathcal{C}$ (having the appropriate dimensions, domains and ranges); all matrices appearing in these complexes are block-matrices with blocks as specified. b_1 and b_2 are billed here as individual objects of $\mathcal{C}$, but they can equally well be taken to be columns of objects provided (the morphism matrix) ϕ remains invertible.

From the previous lemmas we infer that the Khovanov complex of a tangle is homotopy equivalent to a chain complex without loops in the smoothings, and in which every differential is a non-invertible cobordism. In other words, if (Ω, d) is a complex in $\mathcal{C}ob^3_{\bullet/l}$, we can use lemmas 2.1, 2.2, and obtain a homotopy equivalent chain complex (Ω', d') with no loop in its smoothings and no invertible cobordism in its differentials. We say that (Ω', d') is a *reduced complex* of (Ω, d) .

3. The category Kob_o and alternating planar algebras

In this section we introduce an alternating orientation in the objects of $\mathcal{C}ob^3_{\bullet/l}(k)$. This orientation induces an orientation in the cobordisms of this category. These oriented k-strand smoothings and cobordisms form the objects and morphisms in a new category. The composition between cobordisms in this oriented category is defined in the standard way, and it is regarded as a graded category, in the sense of [BN2, Section 6]. We subject out the cobordisms in this oriented category to the relations in (1) and denote it as $\mathcal{C}ob^3_o(k)$. Now we can follow [BN2] and define sequentially the categories, $\mathrm{Mat}(\mathcal{C}ob^3_o(k))$, $\mathrm{Kom}(\mathrm{Mat}(\mathcal{C}ob^3_o(k)))$ and