Response to Referees' comments on the paper "*Homomorphic expansions for knotted trivalent graphs*"

Dror Bar-Natan and Zsuzsanna Dancso

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Reviewer 1

"In this paper, the authors generalize the universal finite type invariant Z^{old} of Knotted Trivalent Graphs (KTG) so that the resulting invariant is "very well-behaved". First, generalize KGT to dotted KGT (dKGT) and extend Z^{old} by substituting the element $\nu^{\pm 1/2}$ to the dots where ν is Z^{old} of the trivial knot.
In my opinion, the idea given in this paper is a good method to handle Z^{old} rather than a generalization of the invariant. The first part of this paper is very lengthy. However, the contents of the latter half show various useful properties of Z^{old}."

—We fully agree with the referee that the point is understanding Z^{old} in a new way rather than extending it to dotted KTGs. To make this more transparent in the paper, we added a sentence to this effect at the end of the next to last paragraph of the introduction.
The 5-page preliminary section is on the long side, but we do believe that it provides information that is necessary for understanding the main part of the paper and hence makes it more readable, especially for non-experts.

Reviewer 2

Questions, remarks, suggestions

Abstract & Introduction: A reference to the work of Cheptea–Le (Comm. Math. Phys. 2007) could be added.
—Done.

P.2, l.-1: "a cyclic ordering of the three edges" $\rightsquigarrow$ "a cyclic ordering of the three half-edges" (... since, technically, there may be some looped edges.)
—Done.

P.3, l.-13: Although it is clear from the context, I can't see where the notation $\mathcal{K}(\Gamma)$ has been set.
—We introduced the notation $\mathcal{K}(\Gamma)$ at the end of the definition of a KTG.

P.10, l.-14: The sentence "(This is because . . . opposite orientation.)" could be moved to l.14, where the fact that $S(\nu) = \nu$ is already used.
—We moved the sentence as suggested.

P.12-13, Proof of Proposition 3.4: Should you not here refer to Lemma 3.2 instead of Theorem 3.3? As for the dotted edge connected sum, you could also say that it is a special case of tree connected sum, which is an operation of dKTG.
—Indeed the reference was wrong. Made the change for edge connected sum.

P.13, proof of Proposition 3.5: Make explicit the algebra $\mathcal{A}(\Gamma)$ where the identity $\Psi^2 = \Psi$ holds.
—We added a comment at the end of the proof.

P.14: Although it is clear from the context, I can't see where the notation $\mathcal{A}_m$ has been set.
—Added the definition in the second paragraph of Section 4.

P.16, Theorem 4.1: What is the relation between the associator Φ produced by Z and the associator from which the construction of Z^{old} starts in [MO]? Is Theorem 4.1 a way to make associators more symmetric?

—Added a Remark after Theorem 4.1.

P.17–20, proof of Theorem 4.1: The maximal trees in the depicted KTGs are not always easy to distinguish from the rest of the graph. It could be helpful to draw them in a thicker way.
—Done.

P.20, bottom figure: The crosses are missing on the tetrahedra.
—Done.

P.21, middle figure: The indication of the vertex orientations is missing on the theta graph with twisted edges.
—Done.

P.22, proof of Theorem 2.1: It seems that the inclusion $\mathcal{I} \subset \mathcal{F}_1$ is not correct because your objects are framed. For instance, the difference $U_0 - U_1$ (where U_k denotes the k-framed unknot) belongs to $\mathcal{I}$ but does not belong to $\mathcal{F}_1$. I would say that, in your definition of the Vassiliev filtration $\mathcal{F}$, you need to include the framing change move.
True. We changed the definiton of $\mathcal{F}$ on page 4.—

P.24, l.7: I don't see why $\nu := Z(\circlearrowleft)$ "is by definition an invertible element" of $\mathcal{A}(\circlearrowleft)$ (...although I agree that one expects Z to have group-like values).
—We removed the words "by definition": we introduced ν on page 7 as the Kontsevich integral of the un-knot. As we mention there, an explicit formula is known for ν, and it is invertible. (Also, the constant term of any value of the Kontsevich integral is 1, hence they are all invertible, but this is not needed here.)

Typos

Note: Page numbers and lines may be off as the file changed. I corrected these for the comments and questions, but not for typos.
P.3, l.2: “obtained from it by “thickening vertices” $\rightsquigarrow$ “obtained from it by “thickening” vertices”
—Done.

P.4, l.14: “by resolutions of of n-singular immersions” $\rightsquigarrow$ “by resolutions of n-singular immersions”
—Done.

P.4, l.-16: “A chord diagram” $\rightsquigarrow$ “A *chord diagram*” (... this is a definition)
—Done.

P.5, l.-18: “$\mathcal{A}(O)$” $\rightsquigarrow$ “$\mathcal{A}(\circlearrowleft)$” (... KTGs, and in particular, knots are oriented in your paper)
—Done.

P.8, l.11: “$c_{d,a} : \mathcal{K}(\Gamma) \to \mathcal{K}(c_{v,a}\Gamma)$” $\rightsquigarrow$ “$c_{d,a} : \mathcal{K}(\Gamma) \to \mathcal{K}(c_{d,a}\Gamma)$”
—Done.

P.8, footnote: “the KTGs γ_1 and γ_2 each embedded” $\rightsquigarrow$ “the KTGs γ_1 and γ_2 are each embedded”
—Done.

P.13, footnote: Some quotation marks are missing at the end of the sentence.
—Done.

P.14, l.7: “involving $\Phi \in \mathcal{A}(\uparrow_3)$ and $R \in \mathcal{A}(\uparrow_2)$” $\rightsquigarrow$ “involving $\Phi \in \mathcal{A}(\uparrow_3)$ and an extra $R \in \mathcal{A}(\uparrow_2)$” (... in constrast with Φ, R has not been discussed yet)
—Done.

P.15, l.3: “made only of only horizontal chords” $\rightsquigarrow$ “made only of horizontal chords”
—Done.

P.16, middle figure: third arrow “$\xrightarrow{\dot{u}}$” $\rightsquigarrow$ “$\xrightarrow{c^2}$”
—Unzip is needed, but indeed c^2 was missing.

P.17, middle figure: third arrow “$\xrightarrow{\dot{u}}$” $\rightsquigarrow$ “$\xrightarrow{c^2}$”
—Unzip is needed, but indeed c^2 was missing.

P.18, top figure: first arrow “$\xrightarrow{c^5 \circ \sharp^2}$” $\rightsquigarrow$ “$\xrightarrow{c^4 \circ \sharp^2}$”
—Done.

P.18, top figure: second arrow “$\xrightarrow{\dot{u}^2}$” $\rightsquigarrow$ “$\xrightarrow{c^4 \circ \dot{u}^2}$”
—Done.

P.19, l.-14: “The Z value graph” $\rightsquigarrow$ “The Z value of the graph”
—Done.

P.21, top figure: “u_e” $\rightsquigarrow$ “$\dot{u}_e$”, “u_f” $\rightsquigarrow$ “$\dot{u}_f$”
—Done.

P.21, l.-14: "$\mathcal{I} = \mathcal{F}_1 \cdot (\mathcal{F}_1)^n$" $\rightsquigarrow$ "$\mathcal{I}^n = \mathcal{F}_1 \cdot (\mathcal{F}_1)^n$"
—The dot here is a period at the end of a sentence, not multiplication. I inserted a word at the beginning of the next sentence to avoid confusion.

P.22, l.5: "along along any tree" $\rightsquigarrow$ "along any tree"
—Done.

P.23, l.13: "that any chord diagram that any chord" $\rightsquigarrow$ "that any chord"
—Done.

P.23, l.15: "The statement of the lemma" $\rightsquigarrow$ "The statement of the corollary"
—Done.

Other changes

Added thanks to referees in acknowledgements.