Pensieve header: The free-Lie meta-crossed-product structure.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-07"];
<< ../Projects/FreeLie/FreeLie.m

λCollect[λ[ω_, μ_]] := λ[ω, Collect[μ, _h, Expand]];
λ /: Series[λ[ω_, μ_], d_Integer] := λ[ω, Collect[μ, _h, MakeLieSeries[d, #] &]];
Randomλ[d_, m_, n_] := Module[{bas},
   bas = AllLyndonWords[{d}, Characters[StringTake["1234567890", m]]];
   Series[λ[0, Expand[Sum[
         h[j] (RandomInteger[{-2, 2}, Length[bas]].bas),
         {j, n}
       ]]], d] // λCollect
   ];
hL[λ_] := Union[Cases[λ, h[s_] :> s, Infinity]];
λ /: λ[ω1_, μ1_] == λ[ω2_, μ2_] :=
  ω1 == ω2 && (And @@ ((D[μ1, h[#]] == D[μ2, h[#]]) & /@ hL[{μ1, μ2}]));

tm[x_, y_, z_][λ[ω_, μ_]] := λ[0, μ // LieMorphism[{⟨x⟩ → ⟨z⟩, ⟨y⟩ → ⟨z⟩}]]];
hm[x_, y_, z_][λ[ω_, μ_]] := λ[0, Plus[
     μ /. {h[x] → 0, h[y] → 0},
     h[z] BCH[D[μ, h[x]], D[μ, h[y]]]
     ]];
hta[x_, y_, z_][λ[ω_, μ_]] := λ[0,
   FixedPoint[LieMorphism[{
       LW[y] → Ad[-D[μ, h[x]]][LW[z]]
      }],
     μ]
   ];
hta[x_, y_][λ_] := λ // hta[x, y, LW["z"]] // LieMorphism[{LW["z"] → LW[y]}];
dm[x_, y_, z_][λ_] := λ // hta[y, x] // tm[x, y, z] // hm[x, y, z];
Rp[x_, y_] := λ[0, h[y] ⟨x⟩];
λ /: λ[ω1_, μ1_] λ[ω2_, μ2_] := λ[ω1 + ω2, μ1 + μ2];

BCH[4]
```

$$\text{LieSeries}\left[⟨x⟩ + ⟨y⟩, \frac{⟨xy⟩}{2}, \frac{⟨xxy⟩}{12} + \frac{⟨xyy⟩}{12}, \frac{⟨xxyy⟩}{24}\right]$$

```
{λ0 = Randomλ[4, 3, 1],
  λ0 // tm[1, 2, 2],
  t1 = λ0 // tm[1, 2, 2] // tm[2, 3, 3],
  λ0 // tm[2, 3, 3],
  t2 = λ0 // tm[2, 3, 3] // tm[1, 3, 3],
  t1 == t2
 } // ColumnForm
```

λ[0, h[1] LieSeries[2 ⟨1⟩ + ⟨2⟩ - 2 ⟨3⟩, -2 ⟨12⟩ - ⟨13⟩ - 2 ⟨23⟩, -2 ⟨112⟩ + 2 ⟨113⟩ - 2 ⟨122⟩ + 2 ⟨1:
λ[0, h[1] LieSeries[3 ⟨2⟩ - 2 ⟨3⟩, -3 ⟨23⟩, ⟨223⟩ - 3 ⟨233⟩, ⟨2223⟩ + 4 ⟨2333⟩]]
λ[0, h[1] LieSeries[⟨3⟩, 0, 0, 0]]
λ[0, h[1] LieSeries[2 ⟨1⟩ - ⟨3⟩, -3 ⟨13⟩, -3 ⟨133⟩, ⟨1113⟩ - ⟨1133⟩]]
λ[0, h[1] LieSeries[⟨3⟩, 0, 0, 0]]
True

```
{λ0 = Randomλ[4, 2, 3],
  λ0 // hm[1, 2, 2],
  t1 = λ0 // hm[1, 2, 2] // hm[2, 3, 3],
  t2 = λ0 // hm[2, 3, 3] // hm[1, 3, 3],
  t1 == t2
 } // ColumnForm
```

$\lambda[0, h[3] \text{LieSeries}[2\langle1\rangle, \langle12\rangle, 2\langle112\rangle + 2\langle122\rangle, \langle1112\rangle + 2\langle1122\rangle] + h[1] \text{LieSeries}[-2\langle1\rangle -$

$\lambda[0, h[3] \text{LieSeries}[2\langle1\rangle, \langle12\rangle, 2\langle112\rangle + 2\langle122\rangle, \langle1112\rangle + 2\langle1122\rangle] + h[2] \text{LieSeries}[-4\langle1\rangle -$

$\lambda[0, h[3] \text{LieSeries}[-2\langle1\rangle - 3\langle2\rangle, 4\langle12\rangle, -6\langle112\rangle + \frac{19\langle122\rangle}{3}, \frac{10\langle1112\rangle}{3} - \frac{47\langle1122\rangle}{6} + \frac{29\langle1222\rangle}{6}]]$

$\lambda[0, h[3] \text{LieSeries}[-2\langle1\rangle - 3\langle2\rangle, 4\langle12\rangle, -6\langle112\rangle + \frac{19\langle122\rangle}{3}, \frac{10\langle1112\rangle}{3} - \frac{47\langle1122\rangle}{6} + \frac{29\langle1222\rangle}{6}]]$

True

```
{t = Series[Rp[1, 3] Rp[2, 4], 4],
  t // hm[3, 4, 3],
  t // hta[3, 2, 5]
 } // ColumnForm
```

$\lambda[0, h[3] \text{LieSeries}[\langle1\rangle, 0, 0, 0] + h[4] \text{LieSeries}[\langle2\rangle, 0, 0, 0]]$

$\lambda[0, h[3] \text{LieSeries}[\langle1\rangle + \langle2\rangle, \frac{\langle12\rangle}{2}, \frac{\langle112\rangle}{12} + \frac{\langle122\rangle}{12}, \frac{\langle1122\rangle}{24}]]$

$\lambda[0, h[3] \text{LieSeries}[\langle1\rangle, 0, 0, 0] + h[4] \text{LieSeries}[\langle5\rangle, \langle15\rangle, \frac{\langle115\rangle}{2}, \frac{\langle1115\rangle}{6}]]$

```
{λ0 = Randomλ[4, 2, 3],
  λ0 // hta[1, 1],
  λ0 // hta[1, 1] // hta[2, 1],
  t1 = λ0 // hta[1, 1] // hta[2, 1] // hm[1, 2, 1],
  t2 = λ0 // hm[1, 2, 1] // hta[1, 1],
  t1 == t2
 } // ColumnForm
```

$\lambda[0, h[1] \text{LieSeries}[0, -2\langle12\rangle, -\langle112\rangle, 2\langle1222\rangle] + h[3] \text{LieSeries}[2\langle2\rangle, \langle12\rangle, -\langle112\rangle - 2\langle12$

$\lambda[0, h[1] \text{LieSeries}[0, -2\langle12\rangle, -\langle112\rangle, -4\langle1122\rangle + 2\langle1222\rangle] + h[3] \text{LieSeries}[2\langle2\rangle, \langle12\rangle, -\langle$

$\lambda[0, h[1] \text{LieSeries}[0, -2\langle12\rangle, -\langle112\rangle + 4\langle122\rangle, -2\langle1222\rangle] + h[3] \text{LieSeries}[2\langle2\rangle, \langle12\rangle, -\langle1$

$\lambda[0, h[3] \text{LieSeries}[2\langle2\rangle, \langle12\rangle, -\langle112\rangle - 4\langle122\rangle, 2\langle1112\rangle + 2\langle1122\rangle + 6\langle1222\rangle] + h[1] \text{LieSeri}$

$\lambda[0, h[3] \text{LieSeries}[2\langle2\rangle, \langle12\rangle, -\langle112\rangle - 4\langle122\rangle, 2\langle1112\rangle + 2\langle1122\rangle + 6\langle1222\rangle] + h[1] \text{LieSeri}$

True

```
{λ0 = Randomλ[3, 4, 4],
  λ0 // dm[1, 2, 1],
  t1 = λ0 // dm[1, 2, 1] // dm[1, 3, 1],
  t2 = λ0 // dm[2, 3, 2] // dm[1, 2, 1],
  t1 == t2
 } // ColumnForm
```

$\lambda[0, h[1] \text{LieSeries}[\langle2\rangle - \langle4\rangle, \langle12\rangle - \langle14\rangle + 2\langle34\rangle, 2\langle112\rangle + \langle113\rangle - 2\langle122\rangle + 2\langle123\rangle - 2\langle124\rangle +$

$\lambda[0, h[1] \text{LieSeries}[\langle1\rangle + 2\langle3\rangle - 2\langle4\rangle, 7\langle13\rangle - \frac{9\langle14\rangle}{2} + 5\langle34\rangle, \frac{133\langle113\rangle}{6} - \frac{139\langle114\rangle}{12} - \frac{23\langle133\rangle}{3} + \frac{97\langle134}{6}$

$\lambda[0, h[1] \text{LieSeries}[\langle1\rangle - 3\langle4\rangle, -7\langle14\rangle, -\frac{69\langle114\rangle}{2} - \frac{155\langle144\rangle}{6}] + h[4] \text{LieSeries}[-2\langle1\rangle + 2\langle4\rangle, -4$

$\lambda[0, h[1] \text{LieSeries}[\langle1\rangle - 3\langle4\rangle, -7\langle14\rangle, -\frac{69\langle114\rangle}{2} - \frac{155\langle144\rangle}{6}] + h[4] \text{LieSeries}[-2\langle1\rangle + 2\langle4\rangle, -4$

True

```
{λ0 = Randomλ[3, 4, 4],
 λ0 // dm[1, 2, 1],
 t1 = λ0 // dm[1, 2, 1] // dm[1, 3, 1],
 t2 = λ0 // dm[2, 3, 2] // dm[1, 2, 1],
 t1 == t2
} // Timing
```

$\Big\{240.771,$

$\Big\{\lambda[0, h[2]$ LieSeries$[-\langle2\rangle + \langle3\rangle - \langle4\rangle, 2\langle13\rangle - 2\langle14\rangle + \langle23\rangle + 2\langle24\rangle + \langle34\rangle, -\langle112\rangle + 2\langle113\rangle +$
$\langle122\rangle - \langle123\rangle + \langle124\rangle + \langle134\rangle - \langle142\rangle - 2\langle143\rangle - \langle144\rangle -$
$\langle224\rangle - \langle234\rangle + 2\langle243\rangle - \langle244\rangle - \langle334\rangle + 2\langle344\rangle] +$
$h[1]$ LieSeries$[-\langle1\rangle + 2\langle2\rangle - 2\langle3\rangle + \langle4\rangle, 2\langle12\rangle + 2\langle13\rangle + \langle14\rangle + 2\langle23\rangle - 2\langle34\rangle,$
$-\langle112\rangle - 2\langle113\rangle + \langle114\rangle - 2\langle122\rangle - \langle123\rangle - \langle124\rangle + 2\langle134\rangle - 2\langle142\rangle +$
$\langle143\rangle + \langle223\rangle + \langle233\rangle - \langle234\rangle - \langle243\rangle - 2\langle244\rangle - \langle334\rangle - 2\langle344\rangle] +$
$h[3]$ LieSeries$[-\langle2\rangle + 2\langle3\rangle + \langle4\rangle, 2\langle12\rangle - 2\langle13\rangle + \langle14\rangle + 2\langle23\rangle,$
$-2\langle112\rangle - \langle113\rangle + \langle114\rangle - \langle122\rangle + 2\langle123\rangle - \langle124\rangle - 2\langle134\rangle + \langle142\rangle + 2\langle143\rangle +$
$2\langle223\rangle - 2\langle224\rangle - \langle233\rangle - 2\langle234\rangle + \langle243\rangle - 2\langle244\rangle - 2\langle334\rangle + \langle344\rangle] +$
$h[4]$ LieSeries$[-\langle1\rangle - 2\langle3\rangle + 2\langle4\rangle, 2\langle12\rangle - \langle13\rangle - \langle14\rangle - \langle23\rangle - \langle24\rangle - 2\langle34\rangle,$
$-\langle112\rangle - 2\langle113\rangle - \langle114\rangle - 2\langle122\rangle - \langle123\rangle - \langle124\rangle + \langle132\rangle - \langle133\rangle - \langle134\rangle - \langle142\rangle -$
$\langle143\rangle - 2\langle144\rangle - \langle223\rangle - \langle224\rangle + 2\langle233\rangle - 2\langle243\rangle + \langle244\rangle + \langle334\rangle - \langle344\rangle]],$

$\lambda\Big[0, h[1]$ LieSeries$\Big[-\langle3\rangle, \dfrac{15\langle13\rangle}{2} - \dfrac{\langle34\rangle}{2}, \dfrac{22\langle113\rangle}{3} + \dfrac{3\langle114\rangle}{2} + \dfrac{7\langle133\rangle}{4} - \dfrac{2\langle134\rangle}{3} +$
$\dfrac{13\langle143\rangle}{6} - \dfrac{11\langle144\rangle}{2} - \dfrac{9\langle334\rangle}{4} + \dfrac{\langle344\rangle}{3}\Big] + h[3]$ LieSeries$[-\langle1\rangle + 2\langle3\rangle + \langle4\rangle,$
$\langle14\rangle, 5\langle113\rangle - 5\langle114\rangle + \langle133\rangle - 5\langle134\rangle - \langle144\rangle - 2\langle334\rangle + \langle344\rangle] +$
$h[4]$ LieSeries$\Big[-\langle1\rangle - 2\langle3\rangle + 2\langle4\rangle, -\langle13\rangle - 3\langle14\rangle - 2\langle34\rangle,$
$-\dfrac{\langle113\rangle}{2} - \dfrac{7\langle114\rangle}{2} + \dfrac{3\langle133\rangle}{2} + \dfrac{3\langle134\rangle}{2} - 2\langle143\rangle - \dfrac{5\langle144\rangle}{2} + \langle334\rangle - \langle344\rangle\Big]\Big],$

$\lambda\Big[0, h[1]$ LieSeries$\Big[\langle4\rangle, \langle14\rangle, -\dfrac{37\langle114\rangle}{6} - 7\langle144\rangle\Big] +$
$h[4]$ LieSeries$\Big[-3\langle1\rangle + 2\langle4\rangle, -4\langle14\rangle, \dfrac{5\langle114\rangle}{2} - \langle144\rangle\Big]\Big],$

$\lambda\Big[0, h[1]$ LieSeries$\Big[\langle4\rangle, \langle14\rangle, -\dfrac{37\langle114\rangle}{6} - 7\langle144\rangle\Big] +$
$h[4]$ LieSeries$\Big[-3\langle1\rangle + 2\langle4\rangle, -4\langle14\rangle, \dfrac{5\langle114\rangle}{2} - \langle144\rangle\Big]\Big],$ True$\Big\}\Big\}$