Anne-Laure Thiel on Categorification of Virtual Braids

June-07-12
5:00 AM

Algebraic Categorification:

A group or algebra $\rightsquigarrow$ $\mathcal{C}$ category (Abelian & monoidal) s.t. $K(\mathcal{C}) \cong A$.

Soergel bimodules ($W = S_n$)

$S_n \circlearrowright \mathbb{Q}[x_1, \ldots x_n]$ set $\deg X_i = 2$

$R = \mathbb{Q}[X_1 \ldots X_{n-1}]$ where $X_i = x_i - x_{i+1}$

For any $H < S_n$ consider the invariants R^H.

In particular, R^{τ_i} where $\tau_i = (i, i+1)$

$B_i = R \otimes_{R^{\tau_i}} R$

$(M[k])_p := M_{p-k}$ deg. shifts.

Def Soergel bimodules are direct summands of direct sums of tensor products of shifted B_i's.

Morphisms: $br_i : B_i \longrightarrow R$ by
$1 \otimes 1 \longrightarrow 1$

$rb_i : R[2] \to B_i$ by

$$1 \longrightarrow X_i \otimes 1 + 1 \otimes X_i$$

Aside R is a free left R^{τ_i} module of rk 2, generated by $\{1, X_i\}$

Thm (Soergel)

* $B_i \otimes_R B_i \cong B_i \oplus B_i\{2\}$
* $B_i \otimes_R B_j \cong B_j \otimes_R B_i \qquad |i-j|>1$
* $$B_i \otimes B_{i+1} \otimes B_i \oplus B_{i+1}[2] \cong$$
$$B_{i+1} \otimes B_i \otimes B_{i+1} \oplus B_i[2]$$

Hecke Alg: $\mathbb{C}[B_n] / \langle \sigma_i^2 = (q^2-1)\sigma_i + q^2 \rangle$

Kazhdan-Lustig generators: $b_i := q^{-1}(1+\sigma_i)$

The relations become:

$$b_i^2 = (q+q^{-1}) b_i$$
$$b_i b_j = b_j b_i \qquad |i-j|>1$$
$$b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i$$

So

$$b_i \leftrightarrow B_i[-1]$$

Categorification of braid groups B_n:

$\sigma_i = q b_i - 1$ so to σ_i we associate the complex

$$F(\sigma_i):\quad 0 \to \underset{-1}{R[2]} \xrightarrow{rb_i} \underset{0}{B_i} \longrightarrow 0$$

to σ_i^{-1} associate

$$F(\sigma_i^{-1}):\quad 0 \longrightarrow \underset{0}{B_i[-2]} \xrightarrow{br_i} \underset{1}{R[-2]} \to 0$$

to 1 associate

$$F(1):\quad 0 \to \underset{0}{R} \to 0$$

to a word $w = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k}$ $\quad \epsilon_i \in \{\pm 1\}$

associate

$$F(w) = F(\sigma_{i_1}^{\epsilon_1}) \otimes \cdots \otimes F(\sigma_{i_k}^{\epsilon_k})$$

Example:

$F(\sigma_i \sigma_i^{-1})$:

$$0 \longrightarrow \underset{-1}{\underset{B_i}{\overset{\shortparallel}{R \otimes_R B_i}}} \xrightarrow{\begin{pmatrix} -1\otimes br_i \\ rb_i \otimes 1 \end{pmatrix}} \underset{0}{\overset{R}{\overset{\shortparallel}{R\otimes R}} \oplus B_i \otimes B_i[-2]} \xrightarrow{\begin{pmatrix} rb_i \otimes 1 & 1\otimes br_i \end{pmatrix}} \underset{1}{\underset{B_i[-2]}{\overset{\shortparallel}{B_i \otimes R[-2]}}} \to 0$$

Some homotopy comes here

$$F(1):\quad 0 \longrightarrow R \longrightarrow 0$$

Thm (Rouquier 2004) if $w = w'$ in B_n

Then $F(w)$ & $F(w')$ are homotopic.

On to virtual braids...

Twisted bi-modules: R_w for $w \in S_n$

R_w is R as a left R-module.

The right action of $a \in R$ on R_w is multiplication by $w(a)$.

So for $b, a \in R$, $p \in R_w$,

$$b \cdot p \cdot a = b p w(a)$$

<u>Lemma</u> * $R_w \otimes_R R_{w'} \cong R_{ww'}$

* $R_w \otimes_{R_{w'}} R \cong R \otimes_{R^{ww'w^{-1}}} R_w$

<u>examples</u> 1. $R_{\tau_i} \otimes R_{\tau_i} \cong R$

2. if $|i-j|>1$,

$$R_{\tau_i} \otimes_R B_j \cong R_{\tau_i} \otimes_{R^{\sigma_j}} R \cong R \otimes_{R^{\tau_j}} R_{\tau_i} \cong$$

$$\cong B_j \otimes_R R_{\tau_i}$$

... to τ_i we associate the complex

$F(\tau_i)$: $0 \to R_{\tau_i} \to 0$

Thm This is a categorification of

VBn.