Adrien Brochier on A Cohomological Construction of Quantization Functors of Lie Bialgebras (Following B. Enriquez)

June-14-12
6:44 AM

Q: Given some $A^{v,hor}(\uparrow ..) \longrightarrow B[[\hbar]]$

$a_{ij} \longrightarrow r_{ij}$

does it extend to

$$vP_n \longrightarrow B[[\hbar]]$$

$$\sigma_{ij} \longrightarrow R_{ij} = 1 + \hbar r_{ij} + \ldots$$

Thm (Leep, BEER). No

CYBE $\leftrightarrow$ 6T QYBE $\leftrightarrow$ R3

Thm (EK) Every classical r-matrix can be quantized: If r satisfies CYBE the $\exists$ $R = 1 + \hbar r + \ldots$ satisfying QYBE

If $\mathfrak{a}$ is a Lie-bialg

Drinfel'd double: $\mathfrak{g} = \mathfrak{a} \bowtie \mathfrak{a}^*$

$r = \sum e_i \otimes e^i$ solves CYBE.

Examples 1. $\mathfrak{a}^*$ commutative: $\mathfrak{g} = I\mathfrak{a}$.

2. If $r + r^{21} = 0 \longrightarrow PfB_n$

A quantization of $(\mathfrak{g}, r)$ is a Hopf-algebra

$$U_\hbar(g) \quad \& \quad R \in (U_\hbar(g))^{\otimes 2}$$

s.t. $U_\hbar(g)/\hbar U_\hbar(g) = U(g)$

and $R^{12,3} = R^{13} \cdot R^{23}$ & $R^{1,23} = R^{12} R^{13}$

$A^u(\uparrow) \longrightarrow A^v(\uparrow)$ $\qquad t = r + r^{21} \in S^2(g)$ invariant.

$$(U(g)[[\hbar]], \Delta_0, e^{t\hbar/2}, \varphi(\hbar t_{12}, \hbar t_{23}))$$

Let $F \in U^{\otimes 2}(g)[[\hbar]]^\times$

$$\Delta_0 \longmapsto F \Delta_0 F^{-1}$$
$$\varphi \longmapsto F^{1,2} F^{12,3} \varphi (F^{2,3} F^{1,23})^{-1}$$
$$e^{t/2} \longrightarrow F e^{t/2} (F^{2,1})^{-1}$$

Want F s.t. $\tilde{d}F = \varphi$

$\rightsquigarrow$ Cohochschild (Cartier homology)

$$U(g)^{\otimes n} \longrightarrow U(g)^{\otimes (n+1)} \longrightarrow \cdots$$

$$d_n(x) = x \otimes 1 + (-1)^{n+1} 1 \otimes x + \sum_{i=1}^{n} (-1)^i (1 \Delta 1) x$$

Enriquez: $U(g)_{univ} = \bigoplus_{n \geq 0} \left(U(F_n)_{\{d_i\}} \otimes U(F_n)_{\{d_i\}} \right)_{S_n}$

where F_n is the free lie algebra on n generators [it is multi-graded]

$\bigcirc_{\Sigma \delta_i}$ means deg $(1, \ldots, 1)$

$\bigcirc_{S_n}$ means co-invariants

Likewise there is $U(\mathfrak{g})^{\otimes n}_{univ}$

Thm (Enriquez) There is a universal Cartier Complex and its homology is Λ.

really $x \otimes 1 \otimes (1 \otimes x)$

Thm (E) 1. The element $r = x \otimes x \in U(\mathfrak{g})^{\otimes 2}_{univ}$ satisfies CYBE

2. $t = r + r^{21}$ extends to an injective algebra map

$$A^{u,hor}(\uparrow_n) \longrightarrow U(\mathfrak{g})^{\otimes 2}_{univ.}$$

We now try to solve the universal twist equation:

$$\sim \quad df_n = \Psi_n - \langle \tilde{d}F \rangle_n$$

$$F' = 1 + F_1 + \cdots + F_{n-1} + f_n \qquad \text{fails.}$$

Instead go for

$$F' = \underbrace{1 + F_1 \dots + F_{n-1}}_{F} + \lambda + F_n$$

$$\lambda \in (\Lambda^2 \mathfrak{g})_{n-1}$$

$$\rightsquigarrow \quad d_2 F_n + CYBE(r, \lambda) = \varphi_n + \mu + d(K)$$

see video.