Pensieve header: Perturbative β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-05\\beta5.1"];
<< betaCalculus.m
```

```
Clear[ħ];
$PerturbativeDegree = 4;
βSimplify[expr_] := Replace[
   Series[Normal[expr], {ħ, 0, $PerturbativeDegree}],
   sd_SeriesData :> MapAt[Expand, sd, 3]
  ];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   βSimplify[μ]
  ];
```

The Knot-Theoretic Equations

```
{
  V0 = βCollect[
     B[ω[ħ c1, ħ c2], α[ħ c1, ħ c2] t[1] h[1] +
       β[ħ c1, ħ c2] t[1] h[2] + γ[ħ c1, ħ c2] t[2] h[1] + δ[ħ c1, ħ c2] t[2] h[2]]
    ] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  C0 = βCollect[B[κ[ħ c1], 0]] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  eqns0 = GroupLikeQ[V0],
  eqns1 = HardR4[V0],
  eqns2 = TwistEq[V0],
  eqns3 = And[(V0 // dη[1]) == B[1, 0], (V0 // dη[2]) == B[1, 0]],
  eqns4 = V0 ** (V0 // dA[1] // dA[2]) == B[1, 0],
  eqns5 = CapEquation[V0, C0],
  eqns6 = (C0 // tη[1]) == B[1, 0],
  eqns7 = (V0 == Rot120[V0]),
  eqns8 = V0 ** (V0 // dS[1] // dS[2]) == R[1, 2]
 } // ColumnForm
```

A very large output was generated. Here is a sample of it:

```
<<1>>
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
eqns = (eqns0 && eqns1 && eqns2 && eqns3 && eqns4 && eqns5 && eqns6 && eqns8);
```

```
Timing[sol = PerturbativeSolveAlways[eqns, ħ, $PerturbativeDegree, {c1, c2}]]
```

$$\left\{0.39,\ \left\{\alpha_{10}\to 0,\ \beta_{10}\to \frac{1}{8},\ \gamma_1\to 0,\ \alpha_1\to \frac{1}{24},\ \delta_1\to 0,\ \beta_1\to \frac{1}{12},\ \kappa_0\to 1,\ \omega_0\to 1,\ \alpha_0\to 0,\ \beta_0\to \frac{1}{2},\right.\right.$$
$$\delta_0\to 0,\ \gamma_0\to 0,\ \alpha_{20}\to 0,\ \beta_{20}\to \frac{1}{24},\ \gamma_2\to 0,\ \alpha_2\to 0,\ \alpha_{11}\to 0,\ \delta_2\to 0,\ \gamma_{10}\to -\frac{1}{24},\ \beta_2\to 0,$$
$$\beta_{11}\to \frac{1}{48},\ \omega_1\to 0,\ \omega_{10}\to 0,\ \alpha_{30}\to 0,\ \beta_{30}\to \frac{1}{64},\ \gamma_3\to 0,\ \alpha_3\to -\frac{1}{240},\ \alpha_{12}\to -\frac{7}{2880},\ \alpha_{21}\to -\frac{7}{2880},$$
$$\delta_3\to 0,\ \beta_3\to -\frac{1}{120},\ \beta_{12}\to -\frac{1}{360},\ \beta_{21}\to \frac{19}{2880},\ \kappa_2\to \frac{1}{48}\left(-1+48\,\kappa_1^2\right),\ \omega_{11}\to -\frac{1}{48},\ \gamma_{20}\to 0,$$
$$\delta_{11}\to 0,\ \omega_2\to 0,\ \omega_{20}\to 0,\ \gamma_{11}\to 0,\ \delta_{20}\to -\frac{1}{24},\ \delta_{10}\to -\frac{1}{12},\ \alpha_{40}\to 0,\ \beta_{40}\to \frac{1}{160},\ \gamma_4\to 0,\ \alpha_4\to 0,$$
$$\alpha_{13}\to 0,\ \alpha_{22}\to 0,\ \alpha_{31}\to 0,\ \delta_4\to 0,\ \beta_4\to 0,\ \beta_{13}\to -\frac{1}{480},\ \beta_{22}\to -\frac{1}{2880},\ \beta_{31}\to \frac{1}{320},\ \gamma_{30}\to \frac{7}{960},$$
$$\gamma_{21}\to \frac{7}{2880},\ \gamma_{12}\to \frac{1}{720},\ \kappa_3\to \frac{1}{16}\,\kappa_1\left(-1+16\,\kappa_1^2\right),\ \omega_3\to 0,\ \omega_{12}\to 0,\ \omega_{21}\to 0,\ \omega_{30}\to 0,\ \omega_{22}\to \frac{17}{5760},$$
$$\gamma_{22}\to 0,\ \gamma_{31}\to 0,\ \gamma_{40}\to 0,\ \delta_{13}\to 0,\ \delta_{22}\to \frac{1}{720},\ \delta_{31}\to \frac{1}{1920},\ \omega_4\to 0,\ \kappa_4\to \frac{13-480\,\kappa_1^2+3840\,\kappa_1^4}{3840},$$
$$\left.\left.\omega_{13}\to \frac{1}{480},\ \omega_{31}\to \frac{1}{480},\ \omega_{40}\to 0,\ \gamma_{13}\to 0,\ \delta_{40}\to -\frac{1}{80},\ \delta_{21}\to \frac{1}{360},\ \delta_{12}\to \frac{1}{360},\ \delta_{30}\to -\frac{19}{960}\right\}\right\}$$

```
{V0, C0} /. sol // ColumnForm
```

$$\begin{pmatrix} 1-\frac{1}{48}\,(c_1\,c_2)\,\hbar^2+\left(\frac{c_1^3\,c_2}{2880}+\frac{17\,c_1^2\,c_2^2}{23\,040}+\frac{c_1\,c_2^3}{2880}\right)\hbar^4+O[\hbar]^5 & h[1] \\ t[1] & \frac{c_2\,\hbar}{24}+\left(-\frac{7\,c_1^2\,c_2}{5760}-\frac{7\,c_1\,c_2^2}{5760}-\frac{c_2^3}{1440}\right)\hbar^3+O[\hbar]^5 \\ t[2] & -\frac{c_1\,\hbar}{24}+\left(\frac{7\,c_1^3}{5760}+\frac{7\,c_1^2\,c_2}{5760}+\frac{c_1\,c_2^2}{1440}\right)\hbar^3+O[\hbar]^5 \end{pmatrix}$$

$$\begin{pmatrix} 1+c_1\,\kappa_1\,\hbar+\frac{1}{96}\,c_1^2\,\left(-1+48\,\kappa_1^2\right)\hbar^2+\frac{1}{96}\,c_1^3\,\kappa_1\,\left(-1+16\,\kappa_1^2\right)\hbar^3+\frac{c_1^4\,\left(13-480\,\kappa_1^2+3840\,\kappa_1^4\right)\hbar^4}{92\,160}+O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
(V0 /. sol) // Rot120 // dη[1] // dP[2 → 1]
```

$$\begin{pmatrix} 1-\frac{c_1\,\hbar}{2}+\frac{7}{48}\,c_1^2\,\hbar^2-\frac{1}{32}\,c_1^3\,\hbar^3+\frac{121\,c_1^4\,\hbar^4}{23\,040}+O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2}+\frac{c_1\,\hbar}{8}-\frac{1}{48}\,c_1^2\,\hbar^2+\frac{1}{384}\,c_1^3\,\hbar^3-\frac{c_1^4\,\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
(V0 /. sol) // Rot120 // dη[1] // dP[2 → 1] // dA[1] // dcap[1]
```

$$\begin{pmatrix} 1-\frac{c_1\,\hbar}{2}+\frac{7}{48}\,c_1^2\,\hbar^2-\frac{1}{32}\,c_1^3\,\hbar^3+\frac{121\,c_1^4\,\hbar^4}{23\,040}+O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
indvars = Union[Flatten[Union[Cases[Last /@ #, ϵ_k_ :> ϵk, Infinity]] & /@ {sol}]]
```

$$\{\kappa_1\}$$

sol

$$\left\{\alpha_{10} \to 0,\ \beta_{10} \to \frac{1}{8},\ \gamma_1 \to 0,\ \alpha_1 \to \frac{1}{24},\ \delta_1 \to 0,\ \beta_1 \to \frac{1}{12},\ \kappa_0 \to 1,\ \omega_0 \to 1,\ \alpha_0 \to 0,\ \beta_0 \to \frac{1}{2},\ \delta_0 \to 0,\right.$$
$$\gamma_0 \to 0,\ \alpha_{20} \to 0,\ \beta_{20} \to \frac{1}{24},\ \gamma_2 \to 0,\ \alpha_2 \to 0,\ \alpha_{11} \to 0,\ \delta_2 \to 0,\ \gamma_{10} \to -\frac{1}{24},\ \beta_2 \to 0,\ \beta_{11} \to \frac{1}{48},$$
$$\omega_1 \to 0,\ \omega_{10} \to 0,\ \alpha_{30} \to 0,\ \beta_{30} \to \frac{1}{64},\ \gamma_3 \to 0,\ \alpha_3 \to -\frac{1}{240},\ \alpha_{12} \to -\frac{7}{2880},\ \alpha_{21} \to -\frac{7}{2880},\ \delta_3 \to 0,$$
$$\beta_3 \to -\frac{1}{120},\ \beta_{12} \to -\frac{1}{360},\ \beta_{21} \to \frac{19}{2880},\ \kappa_2 \to \frac{1}{48}\left(-1 + 48\,\kappa_1^2\right),\ \omega_{11} \to -\frac{1}{48},\ \gamma_{20} \to 0,\ \delta_{11} \to 0,$$
$$\omega_2 \to 0,\ \omega_{20} \to 0,\ \gamma_{11} \to 0,\ \delta_{20} \to -\frac{1}{24},\ \delta_{10} \to -\frac{1}{12},\ \alpha_{40} \to 0,\ \beta_{40} \to \frac{1}{160},\ \gamma_4 \to 0,\ \alpha_4 \to 0,\ \alpha_{13} \to 0,$$
$$\alpha_{22} \to 0,\ \alpha_{31} \to 0,\ \delta_4 \to 0,\ \beta_4 \to 0,\ \beta_{13} \to -\frac{1}{480},\ \beta_{22} \to -\frac{1}{2880},\ \beta_{31} \to \frac{1}{320},\ \gamma_{30} \to \frac{7}{960},\ \gamma_{21} \to \frac{7}{2880},$$
$$\gamma_{12} \to \frac{1}{720},\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-1 + 16\,\kappa_1^2\right),\ \omega_3 \to 0,\ \omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_{22} \to \frac{17}{5760},\ \gamma_{22} \to 0,$$
$$\gamma_{31} \to 0,\ \gamma_{40} \to 0,\ \delta_{13} \to 0,\ \delta_{22} \to \frac{1}{720},\ \delta_{31} \to \frac{1}{1920},\ \omega_4 \to 0,\ \kappa_4 \to \frac{13 - 480\,\kappa_1^2 + 3840\,\kappa_1^4}{3840},$$
$$\left.\omega_{13} \to \frac{1}{480},\ \omega_{31} \to \frac{1}{480},\ \omega_{40} \to 0,\ \gamma_{13} \to 0,\ \delta_{40} \to -\frac{1}{80},\ \delta_{21} \to \frac{1}{360},\ \delta_{12} \to \frac{1}{360},\ \delta_{30} \to -\frac{19}{960}\right\}$$

```
sol1 = Union[
  sol /. Thread[indvars → 0],
  Thread[indvars → 0]
 ]
```

$$\left\{\alpha_0 \to 0,\ \alpha_1 \to \frac{1}{24},\ \alpha_2 \to 0,\ \alpha_3 \to -\frac{1}{240},\ \alpha_4 \to 0,\ \alpha_{10} \to 0,\ \alpha_{11} \to 0,\ \alpha_{12} \to -\frac{7}{2880},\ \alpha_{13} \to 0,\right.$$
$$\alpha_{20} \to 0,\ \alpha_{21} \to -\frac{7}{2880},\ \alpha_{22} \to 0,\ \alpha_{30} \to 0,\ \alpha_{31} \to 0,\ \alpha_{40} \to 0,\ \beta_0 \to \frac{1}{2},\ \beta_1 \to \frac{1}{12},\ \beta_2 \to 0,$$
$$\beta_3 \to -\frac{1}{120},\ \beta_4 \to 0,\ \beta_{10} \to \frac{1}{8},\ \beta_{11} \to \frac{1}{48},\ \beta_{12} \to -\frac{1}{360},\ \beta_{13} \to -\frac{1}{480},\ \beta_{20} \to \frac{1}{24},\ \beta_{21} \to \frac{19}{2880},$$
$$\beta_{22} \to -\frac{1}{2880},\ \beta_{30} \to \frac{1}{64},\ \beta_{31} \to \frac{1}{320},\ \beta_{40} \to \frac{1}{160},\ \gamma_0 \to 0,\ \gamma_1 \to 0,\ \gamma_2 \to 0,\ \gamma_3 \to 0,\ \gamma_4 \to 0,$$
$$\gamma_{10} \to -\frac{1}{24},\ \gamma_{11} \to 0,\ \gamma_{12} \to \frac{1}{720},\ \gamma_{13} \to 0,\ \gamma_{20} \to 0,\ \gamma_{21} \to \frac{7}{2880},\ \gamma_{22} \to 0,\ \gamma_{30} \to \frac{7}{960},$$
$$\gamma_{31} \to 0,\ \gamma_{40} \to 0,\ \delta_0 \to 0,\ \delta_1 \to 0,\ \delta_2 \to 0,\ \delta_3 \to 0,\ \delta_4 \to 0,\ \delta_{10} \to -\frac{1}{12},\ \delta_{11} \to 0,\ \delta_{12} \to \frac{1}{360},$$
$$\delta_{13} \to 0,\ \delta_{20} \to -\frac{1}{24},\ \delta_{21} \to \frac{1}{360},\ \delta_{22} \to \frac{1}{720},\ \delta_{30} \to -\frac{19}{960},\ \delta_{31} \to \frac{1}{1920},\ \delta_{40} \to -\frac{1}{80},\ \kappa_0 \to 1,$$
$$\kappa_1 \to 0,\ \kappa_2 \to -\frac{1}{48},\ \kappa_3 \to 0,\ \kappa_4 \to \frac{13}{3840},\ \omega_0 \to 1,\ \omega_1 \to 0,\ \omega_2 \to 0,\ \omega_3 \to 0,\ \omega_4 \to 0,\ \omega_{10} \to 0,$$
$$\left.\omega_{11} \to -\frac{1}{48},\ \omega_{12} \to 0,\ \omega_{13} \to \frac{1}{480},\ \omega_{20} \to 0,\ \omega_{21} \to 0,\ \omega_{22} \to \frac{17}{5760},\ \omega_{30} \to 0,\ \omega_{31} \to \frac{1}{480},\ \omega_{40} \to 0\right\}$$

V1 = V0 /. sol1

$$\begin{pmatrix} 1-\frac{1}{48}(c_1 c_2)\hbar^2+\left(\frac{c_1^3 c_2}{2880}+\frac{17 c_1^2 c_2^2}{23040}+\frac{c_1 c_2^3}{2880}\right)\hbar^4+O[\hbar]^5 & h[1] \\ t[1] & \frac{c_2\hbar}{24}+\left(-\frac{7 c_1^2 c_2}{5760}-\frac{7 c_1 c_2^2}{5760}-\frac{c_2^3}{1440}\right)\hbar^3+O[\hbar]^5 \quad \frac{1}{2}+\left(\frac{c_1}{8}+ \\ t[2] & -\frac{c_1\hbar}{24}+\left(\frac{7 c_1^3}{5760}+\frac{7 c_1^2 c_2}{5760}+\frac{c_1 c_2^2}{1440}\right)\hbar^3+O[\hbar]^5 \end{pmatrix}$$

C1 = C0 /. sol1

$$\begin{pmatrix} 1-\frac{1}{96}c_1^2\hbar^2+\frac{13 c_1^4\hbar^4}{92160}+O[\hbar]^5 \\ t[1] \end{pmatrix}$$

HardR4[V1]

True

TwistEq[V1]

True

V1 ** (V1 // dA[1] // dA[2])

(1)

CapEquation[V1, C1]

True

Φ1 = Φ[V1]

$$\begin{pmatrix} 1 & h[1] \\ t[1] & \left(\frac{c_2^2 c_3}{1152}+\frac{c_2 c_3^2}{1152}\right)\hbar^3+O[\hbar]^5 \quad \frac{c_3\hbar}{24}+\left(-\frac{c_1^2 c_3}{1440}-\frac{1}{480}c_1 c_2 c_3- \\ t[2] & -\frac{c_3\hbar}{24}+\left(\frac{c_1^2 c_3}{1440}+\frac{c_1 c_2 c_3}{2880}+\frac{7 c_2^2 c_3}{5760}+\frac{c_1 c_2^2}{5760}+\frac{7 c_2 c_3^2}{5760}+\frac{c_3^3}{1440}\right)\hbar^3+O[\hbar]^5 \quad \left(\frac{c_1^2 c_3}{1152}+ \\ t[3] & \frac{c_2\hbar}{24}+\left(-\frac{c_1^2 c_2}{1440}-\frac{7 c_1 c_2^2}{5760}-\frac{7 c_2^3}{5760}-\frac{1}{960}c_1 c_2 c_3-\frac{7 c_2^2 c_3}{5760}-\frac{c_2 c_3^2}{1440}\right)\hbar^3+O[\hbar]^5 \quad -\frac{c_1\hbar}{24}+\left(\frac{c_1^3}{1440}+\frac{7 c_1^2 c_2}{5760}+\frac{7 c_1 c_2^2}{5760}+ \end{pmatrix}$$

Pentagon[Φ1]

True

Hexagon[+1, Φ1]

True

Hexagon[-1, Φ1]

True

Φ1 ** (Φ1 // dP[3, 2, 1])

(1)

Φ1 ** (Φ1 // dS[1] // dS[2] // dS[3])

(1)

R[1, 2, 1 / 2]

$$\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\hbar}{8}+\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3+\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
{R[1, 2, 1 / 2], R[1, 2, 1 / 2] // dS[1] // dS[2]}
```

$$\left\{\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\hbar}{8}+\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3+\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\hbar}{8}+\frac{1}{48}c_1^2\hbar^2+\frac{1}{384}c_1^3\hbar^3+\frac{c_1^4\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}\right\}$$

```
(R[1, 3, 1 / 2] ** R[2, 3, 1 / 2] ** V1) == (V1 ** (R[1, 3, 1 / 2] // dΔ[1, 1, 2]))
```

$$\frac{1}{2}+\hbar\left(\frac{c_1}{8}-\frac{c_2}{4}\right)+\hbar^2\left(\frac{c_1^2}{48}-\frac{5\,c_1\,c_2}{24}-\frac{5\,c_2^2}{48}\right)+\hbar^3\left(\frac{c_1^3}{384}-\frac{5}{64}c_1^2\,c_2-\frac{5}{64}c_1\,c_2^2-\frac{c_2^3}{48}\right)+$$
$$\hbar^4\left(\frac{c_1^4}{3840}-\frac{19}{960}c_1^3\,c_2-\frac{19}{640}c_1^2\,c_2^2-\frac{7}{480}c_1\,c_2^3-\frac{3\,c_2^4}{1280}\right)==\frac{1}{2}+\hbar\left(\frac{c_1}{8}+\frac{c_2}{8}\right)+\hbar^2\left(\frac{c_1^2}{48}+\frac{c_1\,c_2}{24}+\frac{c_2^2}{48}\right)+$$
$$\hbar^3\left(\frac{c_1^3}{384}+\frac{1}{128}c_1^2\,c_2+\frac{1}{128}c_1\,c_2^2+\frac{c_2^3}{384}\right)+\hbar^4\left(\frac{c_1^4}{3840}+\frac{1}{960}c_1^3\,c_2+\frac{1}{640}c_1^2\,c_2^2+\frac{1}{960}c_1\,c_2^3+\frac{c_2^4}{3840}\right)\&\&$$
$$\frac{1}{2}+\hbar\left(\frac{c_1}{2}+\frac{c_2}{8}\right)+\hbar^2\left(\frac{13\,c_1^2}{48}+\frac{c_1\,c_2}{6}+\frac{c_2^2}{48}\right)+\hbar^3\left(\frac{17\,c_1^3}{192}+\frac{3}{32}c_1^2\,c_2+\frac{1}{32}c_1\,c_2^2+\frac{c_2^3}{384}\right)+$$
$$\hbar^4\left(\frac{27\,c_1^4}{1280}+\frac{31}{960}c_1^3\,c_2+\frac{11}{640}c_1^2\,c_2^2+\frac{7\,c_1\,c_2^3}{1920}+\frac{c_2^4}{3840}\right)==\frac{1}{2}+\hbar\left(\frac{c_1}{8}+\frac{c_2}{8}\right)+\hbar^2\left(\frac{c_1^2}{48}+\frac{c_1\,c_2}{24}+\frac{c_2^2}{48}\right)+$$
$$\hbar^3\left(\frac{c_1^3}{384}+\frac{1}{128}c_1^2\,c_2+\frac{1}{128}c_1\,c_2^2+\frac{c_2^3}{384}\right)+\hbar^4\left(\frac{c_1^4}{3840}+\frac{1}{960}c_1^3\,c_2+\frac{1}{640}c_1^2\,c_2^2+\frac{1}{960}c_1\,c_2^3+\frac{c_2^4}{3840}\right)$$

```
V1 ** (V1 // dS[1] // dS[2])
```

$$\begin{pmatrix} 1 & h[2] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 \end{pmatrix}$$

```
R[1, 2]
```

$$\begin{pmatrix} 1 & h[2] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 \end{pmatrix}$$

```
False && Put[{V1, C1, sol}, "SolutionToDegree4-120523.m"]
```

False

```
False && Put[{V1, C1, sol}, "SolutionToDegree6-120523.m"]
```

False

```
False && Put[{V1, C1, sol}, "SolutionToDegree8-120523.m"]
```

False