Pensieve header: Exact β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-05\\beta5.1"];
<< betaCalculus.m
Clear[ℏ]
```

The Knot-Theoretic Equations

```
{
  V0 = βCollect[
    B[ω[ℏ c1, ℏ c2], α[ℏ c1, ℏ c2] t[1] h[1] +
      β[ℏ c1, ℏ c2] t[1] h[2] + γ[ℏ c1, ℏ c2] t[2] h[1] + δ[ℏ c1, ℏ c2] t[2] h[2]]
   ],
  C0 = βCollect[B[κ[ℏ c1], 0]],
  eqns0 = GroupLikeQ[V0, 1 / 2 c1 h[2]],
  eqns1 = HardR4[V0],
  eqns2 = TwistEq[V0],
  eqns3 = And[(V0 // dη[1]) == B[1, 0], (V0 // dη[2]) == B[1, 0]],
  eqns4 = V0 ** (V0 // dA[1] // dA[2]) == B[1, 0],
  eqns5 = CapEquation[V0, C0],
  eqns6 = (C0 // tη[1]) == B[1, 0],
  eqns7 = (V0 == Rot120[V0]),
  eqns8 = V0 ** (V0 // dS[1] // dS[2]) == R[1, 2]
 } // ColumnForm
```

$$\begin{pmatrix} \omega[\hbar c_1, \hbar c_2] & h[1] & h[2] \\ t[1] & \alpha[\hbar c_1, \hbar c_2] & \beta[\hbar c_1, \hbar c_2] \\ t[2] & \gamma[\hbar c_1, \hbar c_2] & \delta[\hbar c_1, \hbar c_2] \end{pmatrix}$$

$$\begin{pmatrix} \kappa[\hbar c_1] \\ t[1] \end{pmatrix}$$

$$1 + \hbar c_1 \alpha[\hbar c_1, \hbar c_2] + \hbar c_2 \gamma[\hbar c_1, \hbar c_2] == 1 \,\&\&\, 1 + \hbar c_1 \beta[\hbar c_1, \hbar c_2] + \hbar c_2 \delta[\hbar c_1, \hbar c_2] == e^{\frac{\hbar c_1}{2}}$$

$$\frac{-e^{\hbar c_2} + e^{\hbar c_1 + \hbar c_2} + \hbar c_1 \beta[\hbar c_1, \hbar c_2] - e^{\hbar c_2} \hbar c_1 \beta[\hbar c_1, \hbar c_2] - e^{\hbar c_2} \hbar c_2 \gamma[\hbar c_1, \hbar c_2] + e^{\hbar c_1 + \hbar c_2} \hbar c_2 \gamma[\hbar c_1, \hbar c_2]}{\hbar c_1} == \frac{-1 + e^{\hbar (c_1 + c_2)}}{\hbar c_1 + \hbar c_2} \,\&\&\, \frac{-1 + e^{\hbar c_2 - \hbar c}}{}$$

$$\omega[\hbar c_1, \hbar c_2] == \omega[\hbar c_2, \hbar c_1] \,\&\&\, \frac{-e^{\frac{\hbar c_1}{2}} c_1 + e^{\frac{1}{2}\hbar (c_1 + c_2)} c_1 + e^{\frac{1}{2}\hbar (c_1 + c_2)} c_2 - e^{\frac{\hbar c_1}{2} + \frac{1}{2}\hbar (c_1 + c_2)} c_2 + e^{\frac{1}{2}\hbar (c_1 + c_2)} \hbar c_1^2 \alpha[\hbar c_1, \hbar c_2] + e^{\frac{1}{2}\hbar (c_1 + c_2)} \hbar}{}$$

$$\omega[0, \hbar c_2] == 1 \,\&\&\, \delta[0, \hbar c_2] == 0 \,\&\&\, \omega[\hbar c_1, 0] == 1 \,\&\&\, \alpha[\hbar c_1, 0] == 0$$

$$\frac{\omega[\hbar c_1, \hbar c_2]^2 + \hbar c_1 \beta[\hbar c_1, \hbar c_2]\, \omega[\hbar c_1, \hbar c_2]^2 + \hbar c_2 \gamma[\hbar c_1, \hbar c_2]\, \omega[\hbar c_1, \hbar c}{1 + \hbar c_1 \alpha[\hbar c_1, \hbar c_2] + \hbar c_1 \beta[\hbar c_1, \hbar c_2] + \hbar^2 c_1^2 \alpha[\hbar c_1, \hbar c_2]\, \beta[\hbar c_1, \hbar c_2] + \hbar c_2 \gamma[\hbar c_1, \hbar c_2] + \hbar^2 c_1 c_2 \beta[\hbar c_1, \hbar c_2]\, \gamma[\hbar c_1, \hbar c_2] + \hbar c_2 \delta[\hbar c_1, \hbar c}$$

$$\frac{\kappa[\hbar (c_1 + c_2)]\, \omega[\hbar c_1, \hbar c_2] + \hbar c_1 \beta[\hbar c_1, \hbar c_2]\, \kappa[\hbar (c_1 + c_2)]\, \omega[\hbar c_1, \hbar c_2] + \hbar c_2 \gamma[\hbar c_1, \hbar c_2]\, \kappa[\hbar}{1 + \hbar c_1 \alpha[\hbar c_1, \hbar c_2] + \hbar c_1 \beta[\hbar c_1, \hbar c_2] + \hbar^2 c_1^2 \alpha[\hbar c_1, \hbar c_2]\, \beta[\hbar c_1, \hbar c_2] + \hbar c_2 \gamma[\hbar c_1, \hbar c_2] + \hbar^2 c_1 c_2 \beta[\hbar c_1, \hbar c_2]\, \gamma[\hbar c_1, \hbar c_2] + \hbar c_2 \delta[\hbar c_1, \hbar c}$$

$$\kappa[0] == 1$$

$$\omega[\hbar c_1, \hbar c_2] == \frac{\omega[\hbar c_2, -\hbar (c_1 + c_2)]}{1 + \hbar c_2 \beta[\hbar c_2, -\hbar (c_1 + c_2)] - \hbar c_1 \delta[\hbar c_2, -\hbar (c_1 + c_2)] - \hbar c_2 \delta[\hbar c_2, -\hbar (c_1 + c_2)]} \,\&\&\, \alpha[\hbar c_1, \hbar c_2] == -\frac{}{-1 - \hbar c_2 \beta[}$$

$$\frac{\omega[-\hbar c_1, -\hbar c_2]\, \omega[\hbar c_1, \hbar c_2] - \hbar c_1 \beta[-\hbar c_1, -\hbar c_2]\, \omega[-\hbar c_1, -\hbar c_2]\, \omega[\hbar c_1, \hbar c_2] - \hbar c_2 \gamma}{1 - \hbar c_1 \alpha[-\hbar c_1, -\hbar c_2] - \hbar c_1 \beta[-\hbar c_1, -\hbar c_2] + \hbar^2 c_1^2 \alpha[-\hbar c_1, -\hbar c_2]\, \beta[-\hbar c_1, -\hbar c_2] - \hbar c_2 \gamma[-\hbar c_1, -\hbar c_2] + \hbar^2 c_1 c_2 \beta[-\hbar c_1, -\hbar c_2]\, \gamma[-\hbar c_1, -\hbar c_2]}$$

```
eqns = FullSimplify[
   (eqns0 && eqns1 && eqns2 && eqns3 && eqns4 && eqns5 && eqns6 && eqns8) /. ℏ → 1];
```

eqns

$$c_1\,\alpha[c_1, c_2] + c_2\,\gamma[c_1, c_2] == 0 \;\&\&\; e^{\frac{c_1}{2}} == 1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2] \;\&\&$$

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{e^{c_2}\,(-1 + e^{c_1})\,(1 + c_2\,\gamma[c_1, c_2])}{c_1} == (-1 + e^{c_2})\,\beta[c_1, c_2] \;\&\&$$

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{(-1 + e^{c_2})\,(1 + c_1\,\beta[c_1, c_2])}{c_2} == e^{c_2}\,(-1 + e^{c_1})\,\gamma[c_1, c_2] \;\&\&$$

$$\omega[c_1, c_2] == \omega[c_2, c_1] \;\&\&\; \frac{1}{c_1\,(c_1 + c_2)} \left(-c_1 + e^{\frac{c_2}{2}} \left((c_1 + c_2)\,(1 + c_1\,\alpha[c_1, c_2] + c_2\,\gamma[c_1, c_2]) - e^{\frac{c_1}{2}}\,c_2\,(1 + (c_1 + c_2)\,\gamma[c_1, c_2])\right)\right) ==$$

$$\delta[c_2, c_1] \;\&\&\; \frac{-1 + e^{\frac{1}{2}\,(c_1+c_2)}\,(1 + (c_1 + c_2)\,\beta[c_1, c_2])}{c_1 + c_2} ==$$

$$\frac{-1 + (-1 + e^{c_1})\,c_2\,\beta[c_2, c_1] + e^{c_1}\,(1 + c_1\,\gamma[c_2, c_1])}{c_1} \;\&\&$$

$$\frac{-1 + e^{\frac{1}{2}\,(c_1+c_2)}\,(1 + (c_1 + c_2)\,\gamma[c_1, c_2])}{c_1 + c_2} == \beta[c_2, c_1] \;\&\&$$

$$\frac{1}{c_2\,(c_1 + c_2)} \left(-c_2 + e^{\frac{c_1}{2}} \left(-e^{\frac{c_2}{2}}\,c_1\,(1 + (c_1 + c_2)\,\beta[c_1, c_2]) + (c_1 + c_2) \left(1 + c_1\,\beta[c_1, c_2] + c_2 \left(2\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\beta[c_2, c_1] + \delta[c_1, c_2]\right)\right)\right)\right) ==$$

$$e^{c_1}\,\alpha[c_2, c_1] \;\&\&\; \omega[0, c_2] == 1 \;\&\&\; \delta[0, c_2] == 0 \;\&\&\; \omega[c_1, 0] == 1 \;\&\&\; \alpha[c_1, 0] == 0 \;\&\&$$

$$\left((1 + c_1\,\beta[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,\omega[c_1, c_2]^2\right) \Big/ \left((1 + c_1\,\alpha[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,(1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2])\right) == 1 \;\&\&$$

$$\kappa[c_1]\,\kappa[c_2] == ((1 + c_1\,\beta[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,\kappa[c_1 + c_2]\,\omega[c_1, c_2]) / ((1 + c_1\,\alpha[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,(1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2])) \;\&\&$$

$$\kappa[0] == 1 \;\&\&\; 1 + ((-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,\omega[-c_1, -c_2]\,\omega[c_1, c_2]) / ((-1 + c_1\,\alpha[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\delta[-c_1, -c_2])) == 0 \;\&\&$$

$$(\alpha[c_1, c_2]\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2]) + c_2\,\beta[-c_1, -c_2]\,(\gamma[-c_1, -c_2] + \gamma[c_1, c_2]) + \alpha[-c_1, -c_2]\,(-1 + c_1\,\beta[-c_1, -c_2] - c_2\,\gamma[c_1, c_2])) / ((-1 + c_1\,\alpha[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])) == 0 \;\&\&$$

$$-\frac{\beta[-c_1, -c_2] + \beta[c_1, c_2]}{-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2]} == \frac{-1 + e^{c_1}}{c_1} \;\&\&$$

$$\frac{\gamma[-c_1, -c_2] + \gamma[c_1, c_2]}{-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2]} == 0 \;\&\&$$

$$((-1 + c_2\,\gamma[-c_1, -c_2])\,(\delta[-c_1, -c_2] + \delta[c_1, c_2]) + c_1\,(\beta[c_1, c_2]\,(\gamma[-c_1, -c_2] - \delta[-c_1, -c_2]) + \beta[-c_1, -c_2]\,(\gamma[-c_1, -c_2] + \delta[c_1, c_2]))) / ((-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\delta[-c_1, -c_2])) == 0$$

```
red1 = Simplify[eqns /. (rule0 = α[x_, y_] :> -y / x γ[x, y])]
```

$$e^{\frac{c_1}{2}} = 1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2]\;\&\&$$

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{e^{c_2}\,(-1 + e^{c_1})\,(1 + c_2\,\gamma[c_1, c_2])}{c_1} = (-1 + e^{c_2})\,\beta[c_1, c_2]\;\&\&$$

$$\frac{1 - e^{c_1+c_2}}{c_1 + c_2} + \frac{(-1 + e^{c_2})\,(1 + c_1\,\beta[c_1, c_2])}{c_2} = e^{c_2}\,(-1 + e^{c_1})\,\gamma[c_1, c_2]\;\&\&$$

$$\omega[c_1, c_2] = \omega[c_2, c_1]\;\&\&\;\frac{-c_1 + e^{\frac{c_2}{2}}\left(c_1 + c_2 - e^{\frac{c_1}{2}}\,c_2\,(1 + (c_1 + c_2)\,\gamma[c_1, c_2])\right)}{c_1\,(c_1 + c_2)} = \delta[c_2, c_1]\;\&\&$$

$$\frac{-1 + e^{\frac{1}{2}(c_1+c_2)}\,(1 + (c_1 + c_2)\,\beta[c_1, c_2])}{c_1 + c_2} = \frac{-1 + (-1 + e^{c_1})\,c_2\,\beta[c_2, c_1] + e^{c_1}\,(1 + c_1\,\gamma[c_2, c_1])}{c_1}\;\&\&$$

$$\frac{-1 + e^{\frac{1}{2}(c_1+c_2)}\,(1 + (c_1 + c_2)\,\gamma[c_1, c_2])}{c_1 + c_2} = \beta[c_2, c_1]\;\&\&$$

$$\frac{1}{c_2}\left(e^{c_1}\,c_1\,\gamma[c_2, c_1] + 1/(c_1 + c_2)\left(-c_2 + e^{\frac{c_1}{2}}\left(-e^{\frac{c_2}{2}}\,c_1\,(1 + (c_1 + c_2)\,\beta[c_1, c_2]) + (c_1 + c_2)\left(1 + c_1\,\beta[c_1, c_2] + c_2\left(2\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\beta[c_2, c_1] + \delta[c_1, c_2]\right)\right)\right)\right)\right) = 0\;\&\&$$

$$\omega[0, c_2] = 1\;\&\&\;\delta[0, c_2] = 0\;\&\&\;\omega[c_1, 0] = 1\;\&\&\;\frac{(1 + c_1\,\beta[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,\omega[c_1, c_2]^2}{1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2]} = 1\;\&\&\;\kappa[c_1]\,\kappa[c_2] = \frac{(1 + c_1\,\beta[c_1, c_2] + c_2\,\gamma[c_1, c_2])\,\kappa[c_1 + c_2]\,\omega[c_1, c_2]}{1 + c_1\,\beta[c_1, c_2] + c_2\,\delta[c_1, c_2]}\;\&\&$$

$$\kappa[0] = 1\;\&\&\;((-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,\omega[-c_1, -c_2]\,\omega[c_1, c_2])\,/\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\delta[-c_1, -c_2]) = 1\;\&\&$$

$$\frac{c_2\,(\gamma[-c_1, -c_2] + \gamma[c_1, c_2])}{c_1\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])} = 0\;\&\&$$

$$-\frac{\beta[-c_1, -c_2] + \beta[c_1, c_2]}{-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2]} = \frac{-1 + e^{c_1}}{c_1}\;\&\&$$

$$\frac{\gamma[-c_1, -c_2] + \gamma[c_1, c_2]}{-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2]} = 0\;\&\&$$

$$((-1 + c_2\,\gamma[-c_1, -c_2])\,(\delta[-c_1, -c_2] + \delta[c_1, c_2]) + c_1\,(\beta[c_1, c_2]\,(\gamma[-c_1, -c_2] - \delta[-c_1, -c_2]) + \beta[-c_1, -c_2]\,(\gamma[-c_1, -c_2] + \delta[c_1, c_2])))\,/\,((-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\gamma[-c_1, -c_2])\,(-1 + c_1\,\beta[-c_1, -c_2] + c_2\,\delta[-c_1, -c_2])) = 0$$

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Length[red1]
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red2 = Simplify[red1 /. (rule1 = δ[x_, y_] :→ (E^(x / 2) - 1 - x β[x, y]) / y)]
```

$$\frac{1-e^{c_1+c_2}}{c_1+c_2}+\frac{e^{c_2}\left(-1+e^{c_1}\right)\left(1+c_2\,\gamma[c_1,c_2]\right)}{c_1}=\left(-1+e^{c_2}\right)\beta[c_1,c_2]\ \&\&$$

$$\frac{1-e^{c_1+c_2}}{c_1+c_2}+\frac{\left(-1+e^{c_2}\right)\left(1+c_1\,\beta[c_1,c_2]\right)}{c_2}=e^{c_2}\left(-1+e^{c_1}\right)\gamma[c_1,c_2]\ \&\&$$

$$\omega[c_1,c_2]=\omega[c_2,c_1]\ \&\&\ \frac{1-e^{\frac{c_2}{2}}+c_2\,\beta[c_2,c_1]+\frac{-c_1+e^{\frac{c_2}{2}}\left(c_1+c_2-e^{\frac{c_1}{2}}c_2\left(1+(c_1+c_2)\,\gamma[c_1,c_2]\right)\right)}{c_1+c_2}}{c_1}=0\ \&\&$$

$$\frac{-1+e^{\frac{1}{2}(c_1+c_2)}\left(1+(c_1+c_2)\,\beta[c_1,c_2]\right)}{c_1+c_2}=\frac{-1+\left(-1+e^{c_1}\right)c_2\,\beta[c_2,c_1]+e^{c_1}\left(1+c_1\,\gamma[c_2,c_1]\right)}{c_1}\ \&\&$$

$$\frac{-1+e^{\frac{1}{2}(c_1+c_2)}\left(1+(c_1+c_2)\,\gamma[c_1,c_2]\right)}{c_1+c_2}=\beta[c_2,c_1]\ \&\&$$

$$\frac{1}{c_2}\left(1/(c_1+c_2)\left(-e^{\frac{1}{2}(c_1+c_2)}c_1^2\,\beta[c_1,c_2]+\left(-1+e^{c_1}\right)c_2\left(1+c_2\,\beta[c_2,c_1]\right)+c_1\left(e^{c_1}-e^{\frac{1}{2}(c_1+c_2)}+c_2\left(-e^{\frac{1}{2}(c_1+c_2)}\beta[c_1,c_2]+\left(-1+e^{c_1}\right)\beta[c_2,c_1]\right)\right)\right)+e^{c_1}c_1\,\gamma[c_2,c_1]\right)=0\ \&\&$$

$$\omega[0,c_2]=1\ \&\&\ \omega[c_1,0]=1\ \&\&\ e^{-\frac{c_1}{2}}\left(1+c_1\,\beta[c_1,c_2]+c_2\,\gamma[c_1,c_2]\right)\omega[c_1,c_2]^2=1\ \&\&$$

$$\kappa[c_1]\,\kappa[c_2]=e^{-\frac{c_1}{2}}\left(1+c_1\,\beta[c_1,c_2]+c_2\,\gamma[c_1,c_2]\right)\kappa[c_1+c_2]\,\omega[c_1,c_2]\ \&\&\ \kappa[0]=1\ \&\&$$

$$1+e^{\frac{c_1}{2}}\left(-1+c_1\,\beta[-c_1,-c_2]+c_2\,\gamma[-c_1,-c_2]\right)\omega[-c_1,-c_2]\,\omega[c_1,c_2]=0\ \&\&$$

$$\frac{c_2\left(\gamma[-c_1,-c_2]+\gamma[c_1,c_2]\right)}{c_1\left(-1+c_1\,\beta[-c_1,-c_2]+c_2\,\gamma[-c_1,-c_2]\right)}=0\ \&\&$$

$$-\frac{\beta[-c_1,-c_2]+\beta[c_1,c_2]}{-1+c_1\,\beta[-c_1,-c_2]+c_2\,\gamma[-c_1,-c_2]}=\frac{-1+e^{c_1}}{c_1}\ \&\&$$

$$\frac{\gamma[-c_1,-c_2]+\gamma[c_1,c_2]}{-1+c_1\,\beta[-c_1,-c_2]+c_2\,\gamma[-c_1,-c_2]}=0\ \&\&$$

$$\left(c_1\left(e^{c_1}\beta[-c_1,-c_2]+\beta[c_1,c_2]\right)+\left(-1+e^{c_1}\right)\left(-1+c_2\,\gamma[-c_1,-c_2]\right)\right)/\left(c_2\left(-1+c_1\,\beta[-c_1,-c_2]+c_2\,\gamma[-c_1,-c_2]\right)\right)=0$$

```
Length[red2]
```

17

$$\text{Solve}\left[\frac{1-e^{c_1+c_2}}{c_1+c_2}+\frac{e^{c_2}\left(-1+e^{c_1}\right)\left(1+c_2\,\gamma[c_1,c_2]\right)}{c_1}=\left(-1+e^{c_2}\right)\beta[c_1,c_2],\ \gamma[c_1,c_2]\right]\ /.\ \{c_1\to x,\ c_2\to y\}$$

$$\Big\{\Big\{\gamma[x,y]\to\left(e^{-y}\left(-x+e^{y}x+e^{y}y-e^{x+y}y-x^2\beta[x,y]+e^{y}x^2\beta[x,y]-x\,y\,\beta[x,y]+e^{y}x\,y\,\beta[x,y]\right)\right)/\left(\left(-1+e^{x}\right)y\,(x+y)\right)\Big\}\Big\}$$

```
red3 =
 FullSimplify[red2 /. (rule2 = γ[x_, y_] :> (e^-y (-x + e^y x + e^y y - e^(x+y) y - x^2 β[x, y] +
          e^y x^2 β[x, y] - x y β[x, y] + e^y x y β[x, y])) / ((-1 + e^x) y (x + y)))]
```

$$\omega[c_1, c_2] == \omega[c_2, c_1]\ \&\&\ \frac{1}{c_1 + c_2}\left(-1 + e^{\frac{1}{2}(c_1+c_2)}\,(1 + (c_1 + c_2)\,\beta[c_1, c_2]) + 1 / ((-1 + e^{c_2})\,c_1)\right.$$
$$\left.(-c_1 + e^{c_2}\,((c_1 + c_2)\,(1 + c_2\,\beta[c_2, c_1]) - e^{c_1}\,c_2\,(1 + (c_1 + c_2)\,\beta[c_2, c_1])))\right) == 0\ \&\&$$

$$\frac{-c_2 + \mathrm{Csch}\left[\frac{c_1}{2}\right] \mathrm{Sinh}\left[\frac{c_2}{2}\right] c_1\,(1 + (c_1 + c_2)\,\beta[c_1, c_2])}{c_2\,(c_1 + c_2)} == \beta[c_2, c_1]\ \&\&$$

$$\frac{1}{c_2\,(c_1 + c_2)} e^{\frac{1}{2}(c_1+c_2)}\left(c_1\,(-1 - (c_1 + c_2)\,\beta[c_1, c_2]) + \mathrm{Csch}\left[\frac{c_2}{2}\right] \mathrm{Sinh}\left[\frac{c_1}{2}\right] c_2\,(1 + (c_1 + c_2)\,\beta[c_2, c_1])\right) ==$$
$$0\ \&\&\ \omega[0, c_2] == 1\ \&\&\ \omega[c_1, 0] == 1\ \&\&$$

$$\frac{e^{-\frac{c_1}{2}-c_2}\,(-1 + e^{c_1+c_2})\,c_1\,(1 + (c_1 + c_2)\,\beta[c_1, c_2])\,\omega[c_1, c_2]^2}{(-1 + e^{c_1})\,(c_1 + c_2)} == 1\ \&\&$$

$$\kappa[c_1]\,\kappa[c_2] == \left(e^{-\frac{c_1}{2}-c_2}\,(-1 + e^{c_1+c_2})\,c_1\,(1 + (c_1 + c_2)\,\beta[c_1, c_2])\,\kappa[c_1 + c_2]\,\omega[c_1, c_2]\right) /$$
$$((-1 + e^{c_1})\,(c_1 + c_2))\ \&\&\ \kappa[0] == 1\ \&\&$$

$$\left((-1 + e^{c_1})\,c_2 + e^{\frac{c_1}{2}}\,(-1 + e^{c_1+c_2})\,c_1^2\,\beta[-c_1, -c_2]\,\omega[-c_1, -c_2]\,\omega[c_1, c_2] +\right.$$
$$\left.c_1\left(-1 + e^{c_1} + e^{\frac{c_1}{2}}\,(-1 + e^{c_1+c_2})\,(-1 + c_2\,\beta[-c_1, -c_2])\,\omega[-c_1, -c_2]\,\omega[c_1, c_2]\right)\right) /$$
$$((-1 + e^{c_1})\,(c_1 + c_2)) == 0\ \&\&$$

$$(e^{-c_2}\,(-1 + e^{c_2})\,(1 + e^{c_1+c_2}\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2]) + (c_1 + c_2)\,\beta[c_1, c_2])) /$$
$$((-1 + e^{c_1+c_2})\,c_1\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2])) == 0\ \&\&$$

$$\frac{(-1 + e^{c_1})\left(-1 - \frac{(c_1+c_2)\,(\beta[-c_1,-c_2]+\beta[c_1,c_2])}{(-1+e^{c_1+c_2})\,(-1+(c_1+c_2)\,\beta[-c_1,-c_2])}\right)}{c_1} == 0\ \&\&$$

$$(e^{-c_2}\,(-1 + e^{c_2})\,(1 + e^{c_1+c_2}\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2]) + (c_1 + c_2)\,\beta[c_1, c_2])) /$$
$$((-1 + e^{c_1+c_2})\,c_2\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2])) == 0\ \&\&$$

$$((-1 + e^{c_1})\,(1 + e^{c_1+c_2}\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2]) + (c_1 + c_2)\,\beta[c_1, c_2])) /$$
$$((-1 + e^{c_1+c_2})\,c_2\,(-1 + (c_1 + c_2)\,\beta[-c_1, -c_2])) == 0$$

```
Length[red3]
```

14

red4 =

$$\text{FullSimplify}\left[\text{red3 /. } \beta[c_2, c_1] \to \frac{-c_2 + \text{Csch}\left[\frac{c_1}{2}\right] \text{Sinh}\left[\frac{c_2}{2}\right] c_1 (1 + (c_1 + c_2)\, \beta[c_1, c_2])}{c_2 (c_1 + c_2)}\right]$$

$$\omega[c_1, c_2] == \omega[c_2, c_1] \,\&\&\, \omega[0, c_2] == 1 \,\&\&\, \omega[c_1, 0] == 1 \,\&\&$$
$$\frac{e^{-\frac{c_1}{2} - c_2} (-1 + e^{c_1+c_2})\, c_1 (1 + (c_1 + c_2)\, \beta[c_1, c_2])\, \omega[c_1, c_2]^2}{(-1 + e^{c_1}) (c_1 + c_2)} == 1 \,\&\&$$
$$\kappa[c_1]\, \kappa[c_2] == \left(e^{-\frac{c_1}{2} - c_2} (-1 + e^{c_1+c_2})\, c_1 (1 + (c_1 + c_2)\, \beta[c_1, c_2])\, \kappa[c_1 + c_2]\, \omega[c_1, c_2]\right) \Big/ ((-1 + e^{c_1}) (c_1 + c_2)) \,\&\&\, \kappa[0] == 1 \,\&\&$$
$$\left((-1 + e^{c_1})\, c_2 + e^{\frac{c_1}{2}} (-1 + e^{c_1+c_2})\, c_1^2\, \beta[-c_1, -c_2]\, \omega[-c_1, -c_2]\, \omega[c_1, c_2] + c_1 \left(-1 + e^{c_1} + e^{\frac{c_1}{2}} (-1 + e^{c_1+c_2}) (-1 + c_2\, \beta[-c_1, -c_2])\, \omega[-c_1, -c_2]\, \omega[c_1, c_2]\right)\right) \Big/ ((-1 + e^{c_1}) (c_1 + c_2)) == 0 \,\&\&$$
$$(e^{-c_2} (-1 + e^{c_2}) (1 + e^{c_1+c_2} (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2]) + (c_1 + c_2)\, \beta[c_1, c_2])) / ((-1 + e^{c_1+c_2})\, c_1 (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2])) == 0 \,\&\&$$
$$\frac{(-1 + e^{c_1}) \left(-1 - \frac{(c_1+c_2) (\beta[-c_1,-c_2]+\beta[c_1,c_2])}{(-1+e^{c_1+c_2}) (-1+(c_1+c_2)\, \beta[-c_1,-c_2])}\right)}{c_1} == 0 \,\&\&$$
$$(e^{-c_2} (-1 + e^{c_2}) (1 + e^{c_1+c_2} (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2]) + (c_1 + c_2)\, \beta[c_1, c_2])) / ((-1 + e^{c_1+c_2})\, c_2 (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2])) == 0 \,\&\&$$
$$((-1 + e^{c_1}) (1 + e^{c_1+c_2} (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2]) + (c_1 + c_2)\, \beta[c_1, c_2])) / ((-1 + e^{c_1+c_2})\, c_2 (-1 + (c_1 + c_2)\, \beta[-c_1, -c_2])) == 0$$

Length[red4]

11

$$\textbf{Series}\left[\left(\frac{\text{Sinh}[x/2]}{x/2}\right)^{-1/4},\ \{x, 0, 4\}\right]$$

$$1 - \frac{x^2}{96} + \frac{13\, x^4}{92160} + O[x]^5$$

```
red5 = FullSimplify[red4 /. (rule4 = {κ[0] → 1, κ[x_] :→ (Sinh[x / 2] / (x / 2))^(-1/4)})]
```

$$\omega[c_1, c_2] == \omega[c_2, c_1] \,\&\&\, \omega[0, c_2] == 1 \,\&\&\, \omega[c_1, 0] == 1 \,\&\&$$

$$\frac{e^{-\frac{c_1}{2}-c_2}\,(-1+e^{c_1+c_2})\,c_1\,(1+(c_1+c_2)\,\beta[c_1, c_2])\,\omega[c_1, c_2]^2}{(-1+e^{c_1})\,(c_1+c_2)} == 1 \,\&\&$$

$$\frac{1}{\sqrt{2}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}} ==$$

$$\frac{1}{-1+e^{c_1}}\,2^{3/4}\,e^{-\frac{c_2}{2}}\,c_1\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{3/4}(1+(c_1+c_2)\,\beta[c_1, c_2])\,\omega[c_1, c_2] \,\&\&$$

$$\left((-1+e^{c_1})\,c_2+e^{\frac{c_1}{2}}\,(-1+e^{c_1+c_2})\,c_1^2\,\beta[-c_1, -c_2]\,\omega[-c_1, -c_2]\,\omega[c_1, c_2]+ c_1\left(-1+e^{c_1}+e^{\frac{c_1}{2}}\,(-1+e^{c_1+c_2})\,(-1+c_2\,\beta[-c_1, -c_2])\,\omega[-c_1, -c_2]\,\omega[c_1, c_2]\right)\right) \Big/ ((-1+e^{c_1})\,(c_1+c_2)) == 0 \,\&\&$$

$$(e^{-c_2}\,(-1+e^{c_2})\,(1+e^{c_1+c_2}\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])+(c_1+c_2)\,\beta[c_1, c_2])) / ((-1+e^{c_1+c_2})\,c_1\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])) == 0 \,\&\&$$

$$\frac{(-1+e^{c_1})\left(-1-\frac{(c_1+c_2)\,(\beta[-c_1, -c_2]+\beta[c_1, c_2])}{(-1+e^{c_1+c_2})\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])}\right)}{c_1} == 0 \,\&\&$$

$$(e^{-c_2}\,(-1+e^{c_2})\,(1+e^{c_1+c_2}\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])+(c_1+c_2)\,\beta[c_1, c_2])) / ((-1+e^{c_1+c_2})\,c_2\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])) == 0 \,\&\&$$

$$((-1+e^{c_1})\,(1+e^{c_1+c_2}\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])+(c_1+c_2)\,\beta[c_1, c_2])) / ((-1+e^{c_1+c_2})\,c_2\,(-1+(c_1+c_2)\,\beta[-c_1, -c_2])) == 0$$

```
Length[red5]
```

10

```
red6 = FullSimplify[red5 /. (rule5 = ω[x_, y_] :→ κ[x + y] κ[x]^-1 κ[y]^-1) /. rule4]
```

$$\left(\sqrt{2}\, e^{-\frac{c_1}{2}-c_2} \left(-1+e^{c_1+c_2}\right) \sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\; c_1 \sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}} \left(1+(c_1+c_2)\,\beta[c_1,c_2]\right)\right) \Big/ \left(\left(-1+e^{c_1}\right)\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\,(c_1+c_2)\right) == 1 \;\&\&\; \frac{1}{\sqrt{2}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}} == \frac{1}{-1+e^{c_1}}$$

$$2\, e^{-\frac{c_2}{2}} \left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4} c_1 \left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4} \sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\left(1+(c_1+c_2)\,\beta[c_1,c_2]\right) \;\&\&$$

$$\left(\left(-1+e^{c_1}\right) c_2 + \left(\sqrt{2}\, e^{\frac{c_1}{2}} \left(-1+e^{c_1+c_2}\right)\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\; c_1^2 \sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\;\beta[-c_1,-c_2]\right) \Big/ \left(\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\right) + c_1\left(-1+e^{c_1}+\left(\sqrt{2}\, e^{\frac{c_1}{2}}\left(-1+e^{c_1+c_2}\right)\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\left(-1+c_2\,\beta[-c_1,-c_2]\right)\right)\Big/\left(\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\right)\right)\right) \Big/ \left(\left(-1+e^{c_1}\right)(c_1+c_2)\right) == 0 \;\&\&$$

$$\left(e^{-c_2}\left(-1+e^{c_2}\right)\left(1+e^{c_1+c_2}\left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)+(c_1+c_2)\,\beta[c_1,c_2]\right)\right) \Big/ \left(\left(-1+e^{c_1+c_2}\right) c_1 \left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)\right) == 0 \;\&\&$$

$$\frac{\left(-1+e^{c_1}\right)\left(-1-\frac{(c_1+c_2)\left(\beta[-c_1,-c_2]+\beta[c_1,c_2]\right)}{\left(-1+e^{c_1+c_2}\right)\left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)}\right)}{c_1} == 0 \;\&\&$$

$$\left(e^{-c_2}\left(-1+e^{c_2}\right)\left(1+e^{c_1+c_2}\left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)+(c_1+c_2)\,\beta[c_1,c_2]\right)\right) \Big/ \left(\left(-1+e^{c_1+c_2}\right) c_2 \left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)\right) == 0 \;\&\&$$

$$\left(\left(-1+e^{c_1}\right)\left(1+e^{c_1+c_2}\left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)+(c_1+c_2)\,\beta[c_1,c_2]\right)\right) \Big/ \left(\left(-1+e^{c_1+c_2}\right) c_2 \left(-1+(c_1+c_2)\,\beta[-c_1,-c_2]\right)\right) == 0$$

```
Length[red6]
```

7

$$\text{Solve}\left[\left(\sqrt{2}\, e^{-\frac{c_1}{2}-c_2}\left(-1+e^{c_1+c_2}\right)\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1 \sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\left(1+(c_1+c_2)\,\beta[c_1,c_2]\right)\right) \Big/ \left(\left(-1+e^{c_1}\right)\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\,(c_1+c_2)\right) == 1,\ \beta[c_1,c_2]\right] \text{ /. } \{c_1\to x,\ c_2\to y\}$$

$$\left\{\left\{\beta[x,y]\to\left(e^{\frac{x}{2}+y}\left(-1+e^{x}\right)\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}\left(1-\frac{\sqrt{2}\, e^{-\frac{x}{2}-y}\left(-1+e^{x+y}\right) x\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}}{\left(-1+e^{x}\right)(x+y)\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}}\right)\right) \Big/ \left(\sqrt{2}\left(-1+e^{x+y}\right) x\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}\right)\right\}\right\}$$

$$\text{FullSimplify}\left[\left(e^{\frac{x}{2}+y}\left(-1+e^{x}\right)\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}\left(1-\frac{\sqrt{2}\, e^{-\frac{x}{2}-y}\left(-1+e^{x+y}\right) x\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}}{\left(-1+e^{x}\right)(x+y)\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}}\right)\right) \Big/ \left(\sqrt{2}\left(-1+e^{x+y}\right) x\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}\right)\right]$$

$$\left(-\sqrt{2}\left(-1+e^{x+y}\right) x\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}+e^{\frac{x}{2}+y}\left(-1+e^{x}\right) y\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}+2\, e^{x+y}\, x\,\text{Sinh}\left[\frac{x}{2}\right]\sqrt{\frac{\text{Sinh}\left[\frac{x+y}{2}\right]}{x+y}}\right) \Big/ \left(\sqrt{2}\left(-1+e^{x+y}\right) x\,(x+y)\sqrt{\frac{\text{Sinh}\left[\frac{x}{2}\right]}{x}}\sqrt{\frac{\text{Sinh}\left[\frac{y}{2}\right]}{y}}\right)$$

```
FullSimplify[
 red6 /. (rule6 = β[x_, y_] :> (-Sqrt[2] (-1 + E^(x+y)) x Sqrt[Sinh[x/2]/x] Sqrt[Sinh[y/2]/y] + E^(x/2+y)
      (-1 + E^x) y Sqrt[Sinh[(x+y)/2]/(x+y)] + 2 E^(x+y) x Sinh[x/2] Sqrt[Sinh[(x+y)/2]/(x+y)]) /
     (Sqrt[2] (-1 + E^(x+y)) x (x + y) Sqrt[Sinh[x/2]/x] Sqrt[Sinh[y/2]/y]))]
```

True

```
FullSimplify[(-Sqrt[2] (-1 + E^(x+y)) x Sqrt[Sinh[x/2]/x] Sqrt[Sinh[y/2]/y] +
     E^(x/2+y) (-1 + E^x) y Sqrt[Sinh[(x+y)/2]/(x+y)] + 2 E^(x+y) x Sinh[x/2] Sqrt[Sinh[(x+y)/2]/(x+y)]) /
   (Sqrt[2] (-1 + E^(x+y)) x (x + y) Sqrt[Sinh[x/2]/x] Sqrt[Sinh[y/2]/y]) /. {x -> 2 x, y -> 2 y}]
```

$$\left(\mathrm{Csch}[x]\sqrt{\frac{\mathrm{Sinh}[x]}{x}}\left(-\left(-1+e^{2\,(x+y)}\right)x\sqrt{\frac{\mathrm{Sinh}[x]}{x}}\sqrt{\frac{\mathrm{Sinh}[y]}{y}}+e^{x+2y}\left(-1+e^{2x}\right)y\sqrt{\frac{\mathrm{Sinh}[x+y]}{x+y}}+2\,e^{2\,(x+y)}\,x\,\mathrm{Sinh}[x]\sqrt{\frac{\mathrm{Sinh}[x+y]}{x+y}}\right)\right)\Big/\left(2\left(-1+e^{2\,(x+y)}\right)(x+y)\sqrt{\frac{\mathrm{Sinh}[y]}{y}}\right)$$

```
unknowns = {α[c1, c2], β[c1, c2], γ[c1, c2], δ[c1, c2], ω[c1, c2], κ[c1]};
sol = Thread[unknowns → FullSimplify[
     unknowns /. rule0 /. rule1 /. rule2 /. rule5 /. rule4 /. rule6
   ]]
```

$$\Big\{\alpha[c_1, c_2] \to -\Bigg(2\,e^{c_1+\frac{c_2}{2}}\, c_2 \Bigg(-2\,\mathrm{Sinh}\Big[\frac{c_1}{2}\Big]\,\mathrm{Sinh}\Big[\frac{1}{2}(c_1+c_2)\Big] + \sqrt{2}\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_1}{2}\big]}{c_1}}\; c_1\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_2}{2}\big]}{c_2}}\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{1}{2}(c_1+c_2)\big]}{c_1+c_2}}\,(c_1+c_2)\Bigg)\Bigg) \Big/ \big((-1+e^{c_1})\,(-1+e^{c_1+c_2})\,c_1\,(c_1+c_2)\big),\ \beta[c_1, c_2] \to$$

$$\Bigg(e^{\frac{c_1}{2}+c_2}\,(-1+e^{c_1})\,c_2\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{1}{2}(c_1+c_2)\big]}{c_1+c_2}} + c_1\Bigg(-\sqrt{2}\,(-1+e^{c_1+c_2})\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_1}{2}\big]}{c_1}}\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_2}{2}\big]}{c_2}} + 2\,e^{c_1+c_2}\,\mathrm{Sinh}\Big[\frac{c_1}{2}\Big]\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{1}{2}(c_1+c_2)\big]}{c_1+c_2}}\Bigg)\Bigg) \Big/$$

$$\Bigg(\sqrt{2}\,(-1+e^{c_1+c_2})\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_1}{2}\big]}{c_1}}\; c_1\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_2}{2}\big]}{c_2}}\,(c_1+c_2)\Bigg),\ \gamma[c_1, c_2] \to$$

$$\Bigg(2\,e^{c_1+\frac{c_2}{2}}\Bigg(-2\,\mathrm{Sinh}\Big[\frac{c_1}{2}\Big]\,\mathrm{Sinh}\Big[\frac{1}{2}(c_1+c_2)\Big] + \sqrt{2}\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_1}{2}\big]}{c_1}}\; c_1\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{c_2}{2}\big]}{c_2}}\,\sqrt{\frac{\mathrm{Sinh}\big[\frac{1}{2}(c_1+c_2)\big]}{c_1+c_2}}\,(c_1+c_2)\Bigg)\Bigg) \Big/ \big((-1+e^{c_1})\,(-1+e^{c_1+c_2})\,(c_1+c_2)\big),$$

$$\delta[c_1, c_2] \to \frac{e^{\frac{c_1}{2}}}{c_2} - \frac{\sqrt{2}\,e^{c_1+c_2}\,\sqrt{\frac{\mathrm{Sinh}[\frac{c_1}{2}]}{c_1}}\; c_1\,\sqrt{\frac{\mathrm{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}}}{(-1+e^{c_1+c_2})\,\sqrt{\frac{\mathrm{Sinh}[\frac{c_2}{2}]}{c_2}}\; c_2} - \frac{1}{c_1+c_2},$$

$$\omega[c_1, c_2] \to \frac{2^{1/4}\left(\frac{\mathrm{Sinh}[\frac{c_1}{2}]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}[\frac{c_2}{2}]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}[\frac{1}{2}(c_1+c_2)]}{c_1+c_2}\right)^{1/4}},$$

$$\kappa[c_1] \to \frac{1}{2^{1/4}\left(\frac{\mathrm{Sinh}[\frac{c_1}{2}]}{c_1}\right)^{1/4}}\Big\}$$

```
V = V0 /. ℏ → 1 /. sol
```

$$\begin{pmatrix} \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \frac{4\,e^{c_1+\frac{c_2}{2}}\,\text{Sinh}\left[\frac{c_1}{2}\right]\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]c_2-2\sqrt{2}\,e^{c_1+\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,c_2\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}-2\sqrt{2}\ \cdots}{c_1^2-e^{c_1}c_1^2-e^{c_1+c_2}c_1^2+e^{2c_1+c_2}c_1^2+c_1c_2-e^{c_1}c_1c_2-e^{c_1+c_2}c_1c_2+e^{2c}\cdots} \\ t[2] & \frac{-4\,e^{c_1+\frac{c_2}{2}}\,\text{Sinh}\left[\frac{c_1}{2}\right]\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]+2\sqrt{2}\,e^{c_1+\frac{c_2}{2}}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\,c_1^2\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}+2\sqrt{2}\,e^{\cdots}}{c_1-e^{c_1}c_1-e^{c_1+c_2}c_1+e^{2c_1+c_2}c_1+c_2-e^{c_1}c_2-e^{c_1+c_2}c_2+e^{2c_1+c_2}\cdots} \end{pmatrix}$$

```
Unprotect[C];
C = C0 /. ℏ → 1 /. sol
```

$$\begin{pmatrix} \frac{1}{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}} \\ t[1] \end{pmatrix}$$

```
False && Put[{V, C, sol}, "ExactSolution-120528.m"]
```

False

```
FullSimplify[R[2, 3] ** R[1, 3] ** V == V ** (R[1, 3] // dΔ[1, 1, 2]) /. ℏ → 1 ]
```

True

```
FullSimplify[V ** Θ[1, 2] == R[1, 2] ** (V // dP[2, 1]) /. ℏ → 1 ]
```

True

```
FullSimplify[V ** (V // dA[1] // dA[2]) == B[1, 0] /. ℏ → 1 ]

FullSimplify[V ** (V // dS[1] // dS[2]) == R[1, 2] /. ℏ → 1 ]

FullSimplify[(V ** (C // dP[12]) // dcap[1] // dcap[2]) ==
   (C * (C // dP[2]) // dcap[1] // dcap[2]) /. ℏ → 1 ]
```