Pensieve header: $\beta$-calculus revision 5.1.

# Utilities

```
ℏ = 1;
βSimplify[expr_] := expr // Together // ExpandDenominator // ExpandNumerator;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, μ_]] := B[
    βSimplify[ω],
    Collect[μ, _h, Collect[#, _t, βSimplify] &]
   ];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] :> s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | c_s_ :> s, Infinity]];
dL[β_] := Union[hL[β], tL[β]];
βForm[B[ω_, μ_]] := Module[
    {tails, heads, mat},
    tails = tL[B[ω, μ]]; heads = hL[B[ω, μ]];
    mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
    PrependTo[mat, t /@ tails];
    mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
    MatrixForm[mat]
   ];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
B /: B[ω1_, μ1_] == B[ω2_, μ2_] := Module[
   {heads, tails},
   tails = tL[{B[ω1, μ1], B[ω2, μ2]}];
   heads = hL[{B[ω1, μ1], B[ω2, μ2]}];
   (ω1 == ω2) && (
     And @@ Flatten[Outer[
        (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
        tails, heads
       ]]
    )
  ]
```

```
PerturbativeSolveAlways[eqs_, h_, deg_Integer, cs_List] := Module[
  {eqns, sol, nsol, d},
  eqns = eqs /. ser_SeriesData :> Normal[ser] /. (lhs_ == rhs_ :> lhs - rhs == 0);
  sol = SolveAlways[eqns /. ℏ → 0, cs];
  If[Length[sol] > 1, Print["Warning: multiple solutions in degree 0"]];
  sol = First@sol;
  nsol = SolveAlways[eqns /. sol /. ℏ^_ :> 0 /. ℏ → 1, cs];
  If[Length[nsol] > 1, Print["Warning: multiple solutions in degree 1"]];
  nsol = First@nsol;
  sol = Join[sol /. nsol, nsol];
  Do[
   nsol = SolveAlways[eqns /. sol /. ℏ^n_ /; n > d :> 0 /. ℏ → 1, cs];
   If[Length[nsol] > 1, Print["Warning: multiple solutions in degree ", d]];
   nsol = First@nsol;
   sol = Join[sol /. nsol, nsol],
   {d, 2, deg}
  ];
  sol
 ]
```

# The Meta-Cross-Product

## The “Tails” meta-group

```
tm[x_, y_, z_][β_] := βCollect[β /. {t[x] → t[z], t[y] → t[z], c_x → c_z, c_y → c_z}];
tΔ[z_, x_, y_][β_] := βCollect[β /. {t[z] → t[x] + t[y], c_z → c_x + c_y}];
tη[x_][β_] := βCollect[(β /. t[x] → 0) /. c_x → 0];
tS[x_][β_] := βCollect[β /. {t[x] → -t[x], c_x → -c_x}];
tA[_][β_] := βCollect[β];
tP[rules___Rule][β_] := βCollect[
   β /. {t[x_] :> t[x /. {rules}], c_x_ :> c_(x /. {rules})}
  ];
```

## The "Heads" meta-group

```
hm[x_, y_, z_][B[ω_, μ_]] := Module[
   {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
   B[ω, M + h[z] (γx + γy + (γx /. t[i_] :→ ℏ c_i) γy)] // βCollect
  ];
hΔ[z_, x_, y_][β_] := βCollect[β /. h[z] → h[x] + h[y]];
hη[x_][β_] := βCollect[β /. h[x] → 0];
hS[x_][B[ω_, μ_]] := Module[{γ},
   γ = 1 + D[μ, h[x]] /. t[s_] :→ ℏ c_s;
   βCollect[B[ω, μ /. h[x] → -h[x] / γ]]
  ];
hA[x_][β_] := hS[x][β];
hP[rules___Rule][β_] := βCollect[β /. h[x_] :→ h[x /. {rules}]];
```

## The TH → HT and HT → TH Swaps

```
thswap[x_, y_][B[ω_, μ_]] := Module[
   {α, β, γ, δ, ϵ},
   α = Coefficient[μ, h[y] t[x]];
   β = D[μ, t[x]] /. h[y] → 0;
   γ = D[μ, h[y]] /. t[x] → 0;
   δ = μ /. h[y] | t[x] → 0;
   ϵ = 1 + ℏ c_x α;
   B[ω * ϵ, Plus[
      α (1 + (γ /. t[i_] :→ ℏ c_i) / ϵ) h[y] t[x],
      β (1 + (γ /. t[i_] :→ ℏ c_i) / ϵ) t[x],
      γ / ϵ h[y],
      δ - ℏ c_x / ϵ γ * β
     ]] // βCollect
  ];
htswap[x_, y_][β_] := β // hS[x] // thswap[y, x] // hS[x];
```

## The “double” meta-group

```
dm[x_, y_, z_][β_] := β // thswap[x, y] // hm[x, y, z] // tm[x, y, z];
dΔ[z_, x_, y_][β_] := β // tΔ[z, x, y] // hΔ[z, x, y];
dS[s_][β_] := β // htswap[s, s] // hS[s] // tS[s];
dA[s_][β_] := β // htswap[s, s] // hA[s] // tA[s];
dη[s_][β_] := β // hη[s] // tη[s];
dcap[s_][β_] := β // htswap[s, s] // hη[s];
dP[rules___][β_] := β // hP[rules] // tP[rules];
dP[pl_List][β_] := Module[
   {σ, len, β1, k},
   len = Length[pl];
   β1 = β // (dP @@ Table[i → σ[i], {i, len}]);
   Do[
    k = pl[[i, 1]];
    β1 = β1 // dP[σ[i] → k];
    Do[
     β1 = β1 // dΔ[k, k, pl[[i, j]]],
     {j, 2, Length[pl[[i]]]}
    ],
    {i, len}
   ];
   β1
  ];
dP[pl___Integer] := dP[IntegerDigits /@ {pl}];
```

## The “external” product

```
B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];
```

## “Braid-Like” operations

```
Unprotect[NonCommutativeMultiply];
β_ ** ν_ := Module[
   {ρ, σ, labels},
   ρ = β * (ν /. {h[s_] :→ h[σ[s]], t[s_] :→ t[σ[s]], c_s_ :→ c_σ[s]});
   labels = Union[Cases[{β, ν}, h[s_] | t[s_] | c_s_ :→ s, Infinity]];
   Do[
    ρ = ρ // dm[s, σ[s], s],
    {s, labels}
   ];
   ρ
  ];
B /: Inverse[B[ω_, μ_]] := Module[
   {ρ = B[1, μ]},
   Do[ρ = ρ // dA[s], {s, dL[ρ]}];
   ReplacePart[ρ, 1 → 1 / ω] // βCollect
  ];
```

## The Group-Like Condition

```
GroupLikeQ[B[_, μ_]] := Module[
   {σ1, σ2, σ3},
   σ1 = ℏ * μ /. t[i_] :→ c_i;
   σ2 = (∂_h[#] σ1) & /@ hL[{σ1}];
   σ3 = (βSimplify[∂_ℏ,ℏ Log[1 + #]] == 0) & /@ σ2;
   And @@ σ3
  ];
GroupLikeQ[B[_, μ_], σ_] := Module[
   {hs, μ1, μ2, σ1},
   hs = hL[{σ, μ}];
   μ1 = ℏ * μ /. t[i_] :→ c_i;
   μ2 = βSimplify[1 + ∂_h[#] μ1] & /@ hs;
   σ1 = βSimplify[Exp[ℏ ∂_h[#] σ]] & /@ hs;
   And @@ MapThread[Equal, {μ2, σ1}]
  ];
```

## The R-Matrix

```
R[x_, y_, p_] := βCollect[B[1, (E^(p ℏ c_x) - 1) / (ℏ c_x) * t[x] h[y]]];
R[x_, y_] := R[x, y, 1];
Ri[x_, y_] := R[x, y, -1];
Θ[x_, y_, p_] := (R[x, x, p / 2] // dΔ[x, x, y]) ** R[x, x, -p / 2] ** R[y, y, -p / 2];
Θ[x_, y_] := Θ[x, y, 1];
Θi[x_, y_] := Θ[x, y, -1];
```

# Testing the meta-cross-product axioms

## The “T” meta-group

```
{
  β = B[ω[c1, c2, c3, c4], Sum[αi[c1, c2, c3, c4] t[i] h[1], {i, 4}]],
  β // tm[1, 2, 1],
  t1 = β // tm[1, 2, 1] // tm[1, 3, 1],
  t2 = β // tm[2, 3, 28] // tm[1, 28, 1],
  t1 == t2
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega[c_1, c_2, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_2, c_3, c_4] \\ t[2] & \alpha_2[c_1, c_2, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_2, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_2, c_3, c_4] \end{pmatrix}$$

$$\begin{pmatrix} \omega[c_1, c_1, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_3, c_4] + \alpha_2[c_1, c_1, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_1, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_3, c_4] \end{pmatrix}$$

$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}$$

$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}$$

True

## The "H" meta-group

```
{
  β = B[ω, Sum[α_{10 i+j} t[i] h[j], {i, 2}, {j, 4}]],
  β // hm[1, 2, 1],
  t1 = β // hm[1, 2, 1] // hm[1, 3, 1],
  t2 = β // hm[2, 3, 28] // hm[1, 28, 1],
  t1 == t2
 } // βForm // ColumnForm
```

| ω | h[1] | h[2] | h[3] | h[4] |
|---|---|---|---|---|
| t[1] | $\alpha_{11}$ | $\alpha_{12}$ | $\alpha_{13}$ | $\alpha_{14}$ |
| t[2] | $\alpha_{21}$ | $\alpha_{22}$ | $\alpha_{23}$ | $\alpha_{24}$ |

| ω | h[1] | h[3] | h[4] |
|---|---|---|---|
| t[1] | $\alpha_{11}+\alpha_{12}+c_1\alpha_{11}\alpha_{12}+c_2\alpha_{12}\alpha_{21}$ | $\alpha_{13}$ | $\alpha_{14}$ |
| t[2] | $\alpha_{21}+\alpha_{22}+c_1\alpha_{11}\alpha_{22}+c_2\alpha_{21}\alpha_{22}$ | $\alpha_{23}$ | $\alpha_{24}$ |

| ω | h[1] |
|---|---|
| t[1] | $\alpha_{11}+\alpha_{12}+c_1\alpha_{11}\alpha_{12}+\alpha_{13}+c_1\alpha_{11}\alpha_{13}+c_1\alpha_{12}\alpha_{13}+c_1^2\alpha_{11}\alpha_{12}\alpha_{13}+c_2\alpha_{12}\alpha_{21}+c_2\alpha_{13}\alpha_{21}+c_1 c_2$ ( |
| t[2] | $\alpha_{21}+\alpha_{22}+c_1\alpha_{11}\alpha_{22}+c_2\alpha_{21}\alpha_{22}+\alpha_{23}+c_1\alpha_{11}\alpha_{23}+c_1\alpha_{12}\alpha_{23}+c_1^2\alpha_{11}\alpha_{12}\alpha_{23}+c_2\alpha_{21}\alpha_{23}+c_1 c_2$ ( |

| ω | h[1] |
|---|---|
| t[1] | $\alpha_{11}+\alpha_{12}+c_1\alpha_{11}\alpha_{12}+\alpha_{13}+c_1\alpha_{11}\alpha_{13}+c_1\alpha_{12}\alpha_{13}+c_1^2\alpha_{11}\alpha_{12}\alpha_{13}+c_2\alpha_{12}\alpha_{21}+c_2\alpha_{13}\alpha_{21}+c_1 c_2$ ( |
| t[2] | $\alpha_{21}+\alpha_{22}+c_1\alpha_{11}\alpha_{22}+c_2\alpha_{21}\alpha_{22}+\alpha_{23}+c_1\alpha_{11}\alpha_{23}+c_1\alpha_{12}\alpha_{23}+c_1^2\alpha_{11}\alpha_{12}\alpha_{23}+c_2\alpha_{21}\alpha_{23}+c_1 c_2$ ( |

True

```
{
  β = B[ω, Sum[α_{10 i+j}[c_1, c_2] * t[i] h[j], {i, 2}, {j, 2}]],
  β // tΔ[2, 2, 3],
  β // hΔ[2, 2, 3],
  β // hΔ[2, 2, 3] // hS[3],
  β // hΔ[2, 2, 3] // hS[3] // hm[2, 3, 2],
  β // hΔ[2, 2, 3] // hS[3] // hm[3, 2, 2],
  β // hS[1],
  β // hS[1] // hS[1]
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2 + c_3] & \alpha_{12}[c_1, c_2 + c_3] \\ t[2] & \alpha_{21}[c_1, c_2 + c_3] & \alpha_{22}[c_1, c_2 + c_3] \\ t[3] & \alpha_{21}[c_1, c_2 + c_3] & \alpha_{22}[c_1, c_2 + c_3] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] & -\frac{\alpha_{12}[c_1,c_2]}{1+c_1\,\alpha_{12}[c_1,c_2]+c_2\,\alpha_{22}[c_1,c_2]} \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] & -\frac{\alpha_{22}[c_1,c_2]}{1+c_1\,\alpha_{12}[c_1,c_2]+c_2\,\alpha_{22}[c_1,c_2]} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & -\frac{\alpha_{11}[c_1,c_2]}{1+c_1\,\alpha_{11}[c_1,c_2]+c_2\,\alpha_{21}[c_1,c_2]} & \alpha_{12}[c_1, c_2] \\ t[2] & -\frac{\alpha_{21}[c_1,c_2]}{1+c_1\,\alpha_{11}[c_1,c_2]+c_2\,\alpha_{21}[c_1,c_2]} & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

```
{
  β = B[ω, Sum[α_{10 i+j} * t[i] h[j], {i, 2}, {j, 3}]],
  t1 = β // hm[1, 2, 1] // hS[1],
  t2 = β // hS[1] // hS[2] // hm[2, 1, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \frac{-\alpha_{11}-\alpha_{12}-c_1\,\alpha_{11}\,\alpha_{12}-c_2\,\alpha_{12}\,\alpha_{21}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \alpha_{13} \\ t[2] & \frac{-\alpha_{21}-\alpha_{22}-c_1\,\alpha_{11}\,\alpha_{22}-c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \frac{-\alpha_{11}-\alpha_{12}-c_1\,\alpha_{11}\,\alpha_{12}-c_2\,\alpha_{12}\,\alpha_{21}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \alpha_{13} \\ t[2] & \frac{-\alpha_{21}-\alpha_{22}-c_1\,\alpha_{11}\,\alpha_{22}-c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \alpha_{23} \end{pmatrix}$$

True

## Testing "thswap"

```
Clear[β];
{β1 = B[ω, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // thswap[1, 1]
 } // βForm
```

$$\left\{\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \end{pmatrix}, \begin{pmatrix} \omega+\alpha\,\omega\,c_1 & h[1] & h[2] \\ t[1] & \frac{\alpha+\alpha^2\,c_1+\alpha\,\gamma\,c_2}{1+\alpha\,c_1} & \frac{\beta+\alpha\,\beta\,c_1+\beta\,\gamma\,c_2}{1+\alpha\,c_1} \\ t[2] & \frac{\gamma}{1+\alpha\,c_1} & \frac{\delta-\beta\,\gamma\,c_1+\alpha\,\delta\,c_1}{1+\alpha\,c_1} \end{pmatrix}\right\}$$

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 3}]],
  β // hm[1, 2, 1],
  t1 = β // hm[1, 2, 1] // thswap[1, 1],
  t2 = β // thswap[1, 1] // thswap[1, 2] // hm[1, 2, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_2 \alpha_{12} \alpha_{21} & \alpha_{13} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{12} + \omega c_1^2 \alpha_{11} \alpha_{12} + \omega c_1 c_2 \alpha_{12} \alpha_{21} & \\ t[1] & \frac{\alpha_{11}+c_1 \alpha_{11}^2+\alpha_{12}+3 c_1 \alpha_{11} \alpha_{12}+2 c_1^2 \alpha_{11}^2 \alpha_{12}+c_1 \alpha_{12}^2+2 c_1^2 \alpha_{11} \alpha_{12}^2+c_1^3 \alpha_{11}^2 \alpha_{12}^2}{} \\ t[2] & \end{pmatrix}$$

$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{12} + \omega c_1^2 \alpha_{11} \alpha_{12} + \omega c_1 c_2 \alpha_{12} \alpha_{21} & \\ t[1] & \frac{\alpha_{11}+c_1 \alpha_{11}^2+\alpha_{12}+3 c_1 \alpha_{11} \alpha_{12}+2 c_1^2 \alpha_{11}^2 \alpha_{12}+c_1 \alpha_{12}^2+2 c_1^2 \alpha_{11} \alpha_{12}^2+c_1^3 \alpha_{11}^2 \alpha_{12}^2}{} \\ t[2] & \end{pmatrix}$$

True

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // thswap[1, 1],
  t2 = β // thswap[2, 1] // thswap[1, 1] // tm[1, 2, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \end{pmatrix}$$

$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{21} & h[1] & h[2] \\ t[1] & \frac{\alpha_{11}+c_1 \alpha_{11}^2+\alpha_{21}+2 c_1 \alpha_{11} \alpha_{21}+c_1 \alpha_{21}^2+c_3 \alpha_{11} \alpha_{31}+c_3 \alpha_{21} \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} & \frac{\alpha_{12}+c_1 \alpha_{11} \alpha_{12}+c_1 \alpha_{12} \alpha_{21}+\alpha_{22}+c_1 \alpha_{11} \alpha_{22}+c_1 \alpha_{21} \alpha_{22}+c}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} & \frac{-c_1 \alpha_{12} \alpha_{31}-c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_1 \alpha_{11} \alpha_{32}+c_1}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} \end{pmatrix}$$

$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{21} & h[1] & h[2] \\ t[1] & \frac{\alpha_{11}+c_1 \alpha_{11}^2+\alpha_{21}+2 c_1 \alpha_{11} \alpha_{21}+c_1 \alpha_{21}^2+c_3 \alpha_{11} \alpha_{31}+c_3 \alpha_{21} \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} & \frac{\alpha_{12}+c_1 \alpha_{11} \alpha_{12}+c_1 \alpha_{12} \alpha_{21}+\alpha_{22}+c_1 \alpha_{11} \alpha_{22}+c_1 \alpha_{21} \alpha_{22}+c}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} & \frac{-c_1 \alpha_{12} \alpha_{31}-c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_1 \alpha_{11} \alpha_{32}+c_1}{1+c_1 \alpha_{11}+c_1 \alpha_{21}} \end{pmatrix}$$

True

## Testing “htswap”

```
Clear[β];
{β1 = B[ω, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // htswap[1, 1]
 } // βForm
```

$$\left\{\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \end{pmatrix}, \begin{pmatrix} \frac{\omega+\gamma\,\omega\,c_2}{1+\alpha\,c_1+\gamma\,c_2} & h[1] & h[2] \\ t[1] & \frac{\alpha}{1+\gamma\,c_2} & \frac{\beta}{1+\gamma\,c_2} \\ t[2] & \frac{\gamma+\alpha\,\gamma\,c_1+\gamma^2\,c_2}{1+\gamma\,c_2} & \frac{\delta+\beta\,\gamma\,c_1+\gamma\,\delta\,c_2}{1+\gamma\,c_2} \end{pmatrix}\right\}$$

```
{
  β = B[ω, Sum[α_{10 i+j} t[i] h[j], {i, 2}, {j, 3}]],
  t1 = β // hm[1, 2, 1] // htswap[1, 1],
  t2 = β // htswap[2, 1] // htswap[1, 1] // hm[1, 2, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega+\omega\,c_2\,\alpha_{21}+\omega\,c_2\,\alpha_{22}+\omega\,c_1\,c_2\,\alpha_{11}\,\alpha_{22}+\omega\,c_2^2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \\ t[1] & \\ t[2] & \frac{\alpha_{21}+c_1\,\alpha_{11}\,\alpha_{21}+c_1\,\alpha_{12}\,\alpha_{21}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{21}^2+c_1\,c_2\,\alpha_{12}\,\alpha_{21}^2+\alpha_2}{} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega+\omega\,c_2\,\alpha_{21}+\omega\,c_2\,\alpha_{22}+\omega\,c_1\,c_2\,\alpha_{11}\,\alpha_{22}+\omega\,c_2^2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{21}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{22}+c_1\,c_2\,\alpha_{11}\,\alpha_{22}+c_2^2\,\alpha_{21}\,\alpha_{22}} & \\ t[1] & \\ t[2] & \frac{\alpha_{21}+c_1\,\alpha_{11}\,\alpha_{21}+c_1\,\alpha_{12}\,\alpha_{21}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{21}^2+c_1\,c_2\,\alpha_{12}\,\alpha_{21}^2+\alpha_2}{} \end{pmatrix}$$

```
True
```

```
{
  β = B[ω, Sum[α_{10 i+j} t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // htswap[1, 1],
  t2 = β // htswap[1, 1] // htswap[1, 2] // tm[1, 2, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

| $\omega$ | h[1] | h[2] |
|---|---|---|
| t[1] | $\alpha_{11}$ | $\alpha_{12}$ |
| t[2] | $\alpha_{21}$ | $\alpha_{22}$ |
| t[3] | $\alpha_{31}$ | $\alpha_{32}$ |

| $\frac{\omega+\omega c_3 \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}+c_3 \alpha_{31}}$ | h[1] | h[2] |
|---|---|---|
| t[1] | $\frac{\alpha_{11}+\alpha_{21}}{1+c_3 \alpha_{31}}$ | $\frac{\alpha_{12}+\alpha_{22}}{1+c_3 \alpha_{31}}$ |
| t[3] | $\frac{\alpha_{31}+c_1 \alpha_{11} \alpha_{31}+c_1 \alpha_{21} \alpha_{31}+c_3 \alpha_{31}^2}{1+c_3 \alpha_{31}}$ | $\frac{c_1 \alpha_{12} \alpha_{31}+c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_3 \alpha_{31} \alpha_{32}}{1+c_3 \alpha_{31}}$ |

| $\frac{\omega+\omega c_3 \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}+c_3 \alpha_{31}}$ | h[1] | h[2] |
|---|---|---|
| t[1] | $\frac{\alpha_{11}+\alpha_{21}}{1+c_3 \alpha_{31}}$ | $\frac{\alpha_{12}+\alpha_{22}}{1+c_3 \alpha_{31}}$ |
| t[3] | $\frac{\alpha_{31}+c_1 \alpha_{11} \alpha_{31}+c_1 \alpha_{21} \alpha_{31}+c_3 \alpha_{31}^2}{1+c_3 \alpha_{31}}$ | $\frac{c_1 \alpha_{12} \alpha_{31}+c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_3 \alpha_{31} \alpha_{32}}{1+c_3 \alpha_{31}}$ |

True

## The "double" meta-group

```
{β = B[ω, Sum[α_{10 i+j} t[i] h[j], {i, 4}, {j, 4}]],
  t1 = β // dm[1, 2, 1] // dm[1, 3, 1],
  t2 = β // dm[2, 3, 2] // dm[1, 2, 1],
  t1 == t2 // Simplify
 } // βForm // ColumnForm
```

A very large output was generated. Here is a sample of it:

| $\omega$ | h[1] | h[2] | h[3] | h[4] |
|---|---|---|---|---|
| t[1] | $\alpha_{11}$ | $\alpha_{12}$ | $\alpha_{13}$ | $\alpha_{14}$ |
| t[2] | $\alpha_{21}$ | $\alpha_{22}$ | $\alpha_{23}$ | $\alpha_{24}$ |
| t[3] | $\alpha_{31}$ | $\alpha_{32}$ | $\alpha_{33}$ | $\alpha_{34}$ |
| t[4] | $\alpha_{41}$ | $\alpha_{42}$ | $\alpha_{43}$ | $\alpha_{44}$ |

| $\omega + \omega c_1 \alpha_{12} + \omega c_1 \alpha_{13} + \omega c_1^2 \alpha_{12} \alpha_{13} + \omega c_1 \alpha_{23} + \omega c_1^2 \alpha_{12} \alpha_{23} + \omega c_1^2 \alpha_{13} \alpha_{32} + \omega c_1 c_4 \alpha_{13} \alpha_{42}$ | |
|---|---|
| t[1] | $\frac{\alpha_{11}+\alpha_{12}+2 c_1 \alpha_1}{1+c_1 \alpha_{12}+c_1 \alpha_{13}}$ |
| t[4] | |

| $\omega + \omega c_1 \alpha_{12} + \omega c_1 \alpha_{13} + \omega c_1^2 \alpha_{12} \alpha_{13} + \omega c_1 \alpha_{23} + \omega c_1^2 \alpha_{12} \alpha_{23} + \omega c_1^2 \alpha_{13} \alpha_{32} + \omega c_1 c_4 \alpha_{13} \alpha_{42}$ | |
|---|---|
| t[1] | $\frac{\alpha_{11}+\alpha_{12}+\cdot}{1+c_1 \alpha_{12}+c_1 \alpha_{13}}$ |
| t[4] | |

True

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### The “braid-like” operations

```
{β = B[ω, Sum[α_{10 i+j}[c1, c2] t[i] h[j], {i, 2}, {j, 2}]],
   Inverse[β],
   β ** Inverse[β]
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\omega} & h[1] \\ t[1] & \frac{-\alpha_{11}[c_1,c_2]-c_1\,\alpha_{11}[c_1,c_2]\,\alpha_{12}[c_1,c_2]-c_2\,\alpha_{12}[c_1,c_2]\,\alpha_{21}[c_1,c_2]}{1+c_1\,\alpha_{11}[c_1,c_2]+c_1\,\alpha_{12}[c_1,c_2]+c_1^2\,\alpha_{11}[c_1,c_2]\,\alpha_{12}[c_1,c_2]+2\,c_2\,\alpha_{21}[c_1,c_2]+c_1\,c_2\,\alpha_{11}[c_1,c_2]\,\alpha_{21}[c_1,c_2]+c_1\,c_2\,\alpha_{12}[c_1,c_2]\,\alpha_{21}[c_1,c_2} \\ t[2] & -\frac{\alpha_{21}[c_1,c_2]}{1+c_1\,\alpha_{12}[c_1,c_2]+c_2\,\alpha_{21}[c_1,c_2]} \end{pmatrix}$$

( 1 )

### The Group-Like Property

```
GroupLikeQ[R[1, 2]]
```

$O[\hbar]^2 == 0$

---

## Some Knot-Theoretic Definitions

```
HardR4[V_] := (R[2, 3] ** R[1, 3] ** V) == (V ** (R[1, 3] // dΔ[1, 1, 2]));
TwistEq[V_] := V ** Θ[1, 2] == R[1, 2] ** (V // dP[2, 1]);
CapEquation[V_, Cap_] := (V ** (Cap // dP[12]) // dcap[1] // dcap[2]) ==
    (Cap (Cap // dP[2]) // dcap[1] // dcap[2]);
Φ[V_] := (Inverse[V] // dP[12, 3]) ** Inverse[V] **
    (V // dP[2, 3]) ** (V // dP[1, 23]);
Pentagon[Φ_] := Φ ** (Φ // dP[1, 23, 4]) ** (Φ // dP[2, 3, 4]) ==
    (Φ // dP[12, 3, 4]) ** (Φ // dP[1, 2, 34]);
Hexagon[s_, Φ_] := Equal[
   Θ[1, 2, s] // dP[12, 3],
   Φ ** Θ[2, 3, s] ** Inverse[Φ // dP[1, 3, 2]] ** Θ[1, 3, s] ** (Φ // dP[3, 1, 2])
  ];
Rot120[β_] := β // dS[2] // dΔ[2, 2, 3] // dm[1, 3, 1] // dP[2, 1];
```

```
{β = B[ω[c1, c2], Sum[α10 i+j[c1, c2] t[i] h[j], {i, 2}, {j, 2}]],
   β // Rot120,
   β // Rot120 // Rot120,
   β // Rot120 // Rot120 // Rot120
  } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega[c_1, c_2] & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega[c_2,-c_1-c_2]}{1+c_2\,\alpha_{12}[c_2,-c_1-c_2]-c_1\,\alpha_{22}[c_2,-c_1-c_2]-c_2\,\alpha_{22}[c_2,-c_1-c_2]} & h[1] & \cdots \\ t[1] & -\frac{\alpha_{22}[c_2,-c_1-c_2]}{-1-c_2\,\alpha_{12}[c_2,-c_1-c_2]+c_1\,\alpha_{22}[c_2,-c_1-c_2]+c_2\,\alpha_{22}[c_2,-c_1-c_2]} & \frac{\cdots}{-1-c_2\,\alpha_{1\cdots}} \\ t[2] & \frac{-\alpha_{12}[c_2,-c_1-c_2]+\alpha_{22}[c_2,-c_1-c_2]}{1+c_2\,\alpha_{12}[c_2,-c_1-c_2]-c_1\,\alpha_{22}[c_2,-c_1-c_2]-c_2\,\alpha_{22}[c_2,-c_1-c_2]} & \frac{\alpha_{11}[c_2,-c_1\cdots}{1+c_2\,\alpha_{12}\cdots} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\omega[-c_1-c_2,c_1]}{-1+c_1\,\alpha_{11}[-c_1-c_2,c_1]+c_2\,\alpha_{11}[-c_1-c_2,c_1]-c_1\,\alpha_{21}[-c_1-c_2,c_1]} & h[1] & \cdots \\ t[1] & \frac{-\alpha_{11}[-c_1-c_2,c_1]+\alpha_{12}[-c_1-c_2,c_1]+\alpha_{21}[-c_1-c_2,c_1]-\alpha_{22}[-c_1-c_2,c_1]}{-1+c_1\,\alpha_{11}[-c_1-c_2,c_1]+c_2\,\alpha_{11}[-c_1-c_2,c_1]-c_1\,\alpha_{21}[-c_1-c_2,c_1]} & \frac{\cdots}{-1\cdots} \\ t[2] & \frac{-\alpha_{11}[-c_1-c_2,c_1]+\alpha_{12}[-c_1-c_2,c_1]}{-1+c_1\,\alpha_{11}[-c_1-c_2,c_1]+c_2\,\alpha_{11}[-c_1-c_2,c_1]-c_1\,\alpha_{21}[-c_1-c_2,c_1]} & -\frac{\cdots}{-\cdots} \end{pmatrix}$$

$$\begin{pmatrix} \omega[c_1, c_2] & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$