Pensieve Header: Comparing the perturbative and the exact solutions.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-05\\beta5.1"];
<< betaCalculus.m
Clear[ℏ]; Unprotect[C];
$PerturbativeDegree = 8;
PExpand[expr_] := expr /. sd_SeriesData :→ MapAt[Expand, sd, 3];
```

```
{V, C, sol} = Get["ExactSolution-120528.m"]
```

$$\left\{\begin{array}{ll} \dfrac{2^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}} & h[1] \\ t[1] & \dfrac{4\, e^{c_1+\frac{c_2}{2}}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right] c_2 - 2\sqrt{2}\, e^{c_1+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\, c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} - 2\sqrt{2}\, e\cdots}{c_1^2 - e^{c_1}c_1^2 - e^{c_1+c_2}c_1^2 + e^{2c_1+c_2}c_1^2 + c_1c_2 - e^{c_1}c_1c_2 - e^{c_1+c_2}c_1c_2 + e^{2c_1+}\cdots} \\ t[2] & \dfrac{-4\, e^{c_1+\frac{c_2}{2}}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right] + 2\sqrt{2}\, e^{c_1+\frac{c_2}{2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1^2\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + 2\sqrt{2}\, e^{c_1}\cdots}{c_1 - e^{c_1}c_1 - e^{c_1+c_2}c_1 + e^{2c_1+c_2}c_1 + c_2 - e^{c_1}c_2 - e^{c_1+c_2}c_2 + e^{2c_1+c_2}} \end{array}\right.$$

$$\Big\{\alpha[c_1, c_2] \to -\left(2\, e^{c_1+\frac{c_2}{2}}\, c_2\left(-2\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right] + \sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\,(c_1+c_2)\right)\right)\Big/\left((-1+e^{c_1})(-1+e^{c_1+c_2})\, c_1\,(c_1+c_2)\right),$$

$$\beta[c_1, c_2] \to \left(e^{\frac{c_1}{2}+c_2}(-1+e^{c_1})\, c_2\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}} + c_1\left(-\sqrt{2}\,(-1+e^{c_1+c_2})\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}} + 2\, e^{c_1+c_2}\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\sqrt{\frac{\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}\right)\right)\Big/$$

$$\left(\sqrt{2}\,(-1+e^{c_1+c_2})\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\,(c_1+c_2)\right),\ \gamma[c_1, c_2] \to$$

$$\left(2\, e^{c_1+\frac{c_2}{2}}\left(-2\,\mathrm{Sinh}\left[\frac{c_1}{2}\right]\mathrm{Sinh}\left[\frac{1}{2}(c_1+c_2)\right] + \sqrt{2}\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\, c_1\sqrt{\frac{\mathrm{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\right.\right.$$

$$\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}\left(c_1+c_2\right)\right]}{c_1+c_2}}\ \left(c_1+c_2\right)\Bigg)\Bigg) \Big/ \left(\left(-1+e^{c_1}\right)\left(-1+e^{c_1+c_2}\right)\left(c_1+c_2\right)\right),$$

$$\delta[c_1, c_2] \to \frac{e^{\frac{c_1}{2}}}{c_2} - \frac{\sqrt{2}\ e^{c_1+c_2}\sqrt{\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}}\ c_1\sqrt{\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}}}{\left(-1+e^{c_1+c_2}\right)\sqrt{\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}}\ c_2} - \frac{1}{c_1+c_2},$$

$$\omega[c_1, c_2] \to \frac{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_2}{2}\right]}{c_2}\right)^{1/4}}{\left(\frac{\text{Sinh}\left[\frac{1}{2}(c_1+c_2)\right]}{c_1+c_2}\right)^{1/4}},$$

$$\kappa[c_1] \to \frac{1}{2^{1/4}\left(\frac{\text{Sinh}\left[\frac{c_1}{2}\right]}{c_1}\right)^{1/4}}\}\}$$

V1 = PExpand[Series[# /. {c₁ → ℏ c₁, c₂ → ℏ c₂}, {ℏ, 0, 8}] & /@ V]

$$\Bigg(1 - \frac{1}{48}(c_1 c_2)\hbar^2 + \left(\frac{c_1^3 c_2}{2880} + \frac{17 c_1^2 c_2^2}{23040} + \frac{c_1 c_2^3}{2880}\right)\hbar^4 + \left(-\frac{c_1^5 c_2}{120960} - \frac{c_1^4 c_2^2}{35840} - \frac{103 c_1^3 c_2^3}{2580480} - \frac{c_1^2 c_2^4}{35840} - \frac{c_1 c_2^5}{120960}\right)\hbar^6 + \left(\frac{c_1^7 c_2}{4838400} + \right.$$

t[1]

t[2]

C1 = PExpand[Series[# /. {c₁ → ℏ c₁, c₂ → ℏ c₂}, {ℏ, 0, 8}] & /@ C]

$$\left(1 - \frac{1}{96}c_1^2\hbar^2 + \frac{13 c_1^4 \hbar^4}{92160} - \frac{17 c_1^6 \hbar^6}{6881280} + \frac{167 c_1^8 \hbar^8}{3397386240} + O[\hbar]^9\right)$$

t[1]

{V2, C2, sol2} = Get["SolutionToDegree8-120524.m"]

$$\Bigg\{\Bigg(1 - \frac{1}{48}(c_1 c_2)\hbar^2 + \left(\frac{c_1^3 c_2}{2880} + \frac{17 c_1^2 c_2^2}{23040} + \frac{c_1 c_2^3}{2880}\right)\hbar^4 + \left(-\frac{c_1^5 c_2}{120960} - \frac{c_1^4 c_2^2}{35840} - \frac{103 c_1^3 c_2^3}{2580480} - \frac{c_1^2 c_2^4}{35840} - \frac{c_1 c_2^5}{120960}\right)\hbar^6 + \left(\frac{c_1^7 c_2}{4838400}\right.$$

t[1]

t[2]

$$\alpha_{10} \to 0,\ \alpha_{11} \to 0,\ \alpha_{12} \to -\frac{7}{2880},\ \alpha_{13} \to 0,\ \alpha_{14} \to \frac{13}{10080},\ \alpha_{15} \to 0,\ \alpha_{16} \to -\frac{19}{13440},\ \alpha_{17} \to 0,$$

$$\alpha_{20} \to 0,\ \alpha_{21} \to -\frac{7}{2880},\ \alpha_{22} \to 0,\ \alpha_{23} \to \frac{83}{80640},\ \alpha_{24} \to 0,\ \alpha_{25} \to -\frac{271}{241920},\ \alpha_{26} \to 0,\ \alpha_{30} \to 0,$$

$$\alpha_{31} \to 0,\ \alpha_{32} \to \frac{31}{40320},\ \alpha_{33} \to 0,\ \alpha_{34} \to -\frac{2893}{3225600},\ \alpha_{35} \to 0,\ \alpha_{40} \to 0,\ \alpha_{41} \to \frac{31}{40320},$$

$$\alpha_{42} \to 0,\ \alpha_{43} \to -\frac{2399}{3225600},\ \alpha_{44} \to 0,\ \alpha_{50} \to 0,\ \alpha_{51} \to 0,\ \alpha_{52} \to -\frac{127}{215040},\ \alpha_{53} \to 0,\ \alpha_{60} \to 0,$$

$$\alpha_{61} \to -\frac{127}{215040},\ \alpha_{62} \to 0,\ \alpha_{70} \to 0,\ \alpha_{71} \to 0,\ \alpha_{80} \to 0,\ \beta_0 \to \frac{1}{2},\ \beta_1 \to \frac{1}{12},\ \beta_2 \to 0,\ \beta_3 \to -\frac{1}{120},$$

$$\beta_4 \to 0,\ \beta_5 \to \frac{1}{252},\ \beta_6 \to 0,\ \beta_7 \to -\frac{1}{240},\ \beta_8 \to 0,\ \beta_{10} \to \frac{1}{8},\ \beta_{11} \to \frac{1}{48},\ \beta_{12} \to -\frac{1}{360},\ \beta_{13} \to -\frac{1}{480},$$

$\beta_{14}\to\frac{1}{630},\ \beta_{15}\to\frac{1}{1008},\ \beta_{16}\to-\frac{1}{560},\ \beta_{17}\to-\frac{1}{960},\ \beta_{20}\to\frac{1}{24},\ \beta_{21}\to\frac{19}{2880},\ \beta_{22}\to-\frac{1}{2880},$
$\beta_{23}\to\frac{11}{40320},\ \beta_{24}\to\frac{1}{3360},\ \beta_{25}\to-\frac{53}{60480},\ \beta_{26}\to-\frac{1}{2688},\ \beta_{30}\to\frac{1}{64},\ \beta_{31}\to\frac{1}{320},\ \beta_{32}\to\frac{53}{80640},$
$\beta_{33}\to-\frac{1}{10752},\ \beta_{34}\to-\frac{167}{201600},\ \beta_{35}\to-\frac{1}{7680},\ \beta_{40}\to\frac{1}{160},\ \beta_{41}\to\frac{17}{10080},\ \beta_{42}\to\frac{1}{8064},$
$\beta_{43}\to-\frac{2347}{3225600},\ \beta_{44}\to-\frac{37}{230400},\ \beta_{50}\to\frac{1}{384},\ \beta_{51}\to\frac{1}{1792},\ \beta_{52}\to-\frac{97}{215040},\ \beta_{53}\to-\frac{7}{46080},$
$\beta_{60}\to\frac{1}{896},\ \beta_{61}\to-\frac{11}{215040},\ \beta_{62}\to-\frac{1}{15360},\ \beta_{70}\to\frac{1}{2048},\ \beta_{71}\to\frac{1}{9216},\ \beta_{80}\to\frac{1}{4608},\ \gamma_0\to 0,$
$\gamma_1\to 0,\ \gamma_2\to 0,\ \gamma_3\to 0,\ \gamma_4\to 0,\ \gamma_5\to 0,\ \gamma_6\to 0,\ \gamma_7\to 0,\ \gamma_8\to 0,\ \gamma_{10}\to-\frac{1}{24},\ \gamma_{11}\to 0,$
$\gamma_{12}\to\frac{1}{720},\ \gamma_{13}\to 0,\ \gamma_{14}\to-\frac{1}{2520},\ \gamma_{15}\to 0,\ \gamma_{16}\to\frac{1}{3360},\ \gamma_{17}\to 0,\ \gamma_{20}\to 0,\ \gamma_{21}\to\frac{7}{2880},$
$\gamma_{22}\to 0,\ \gamma_{23}\to-\frac{13}{20160},\ \gamma_{24}\to 0,\ \gamma_{25}\to\frac{19}{40320},\ \gamma_{26}\to 0,\ \gamma_{30}\to\frac{7}{960},\ \gamma_{31}\to 0,\ \gamma_{32}\to-\frac{83}{80640},$
$\gamma_{33}\to 0,\ \gamma_{34}\to\frac{271}{403200},\ \gamma_{35}\to 0,\ \gamma_{40}\to 0,\ \gamma_{41}\to-\frac{31}{20160},\ \gamma_{42}\to 0,\ \gamma_{43}\to\frac{2893}{3225600},$
$\gamma_{44}\to 0,\ \gamma_{50}\to-\frac{31}{8064},\ \gamma_{51}\to 0,\ \gamma_{52}\to\frac{2399}{1935360},\ \gamma_{53}\to 0,\ \gamma_{60}\to 0,\ \gamma_{61}\to\frac{127}{71680},\ \gamma_{62}\to 0,$
$\gamma_{70}\to\frac{127}{30720},\ \gamma_{71}\to 0,\ \gamma_{80}\to 0,\ \delta_0\to 0,\ \delta_1\to 0,\ \delta_2\to 0,\ \delta_3\to 0,\ \delta_4\to 0,\ \delta_5\to 0,\ \delta_6\to 0,$
$\delta_7\to 0,\ \delta_8\to 0,\ \delta_{10}\to-\frac{1}{12},\ \delta_{11}\to 0,\ \delta_{12}\to\frac{1}{360},\ \delta_{13}\to 0,\ \delta_{14}\to-\frac{1}{1260},\ \delta_{15}\to 0,\ \delta_{16}\to\frac{1}{1680},$
$\delta_{17}\to 0,\ \delta_{20}\to-\frac{1}{24},\ \delta_{21}\to\frac{1}{360},\ \delta_{22}\to\frac{1}{720},\ \delta_{23}\to-\frac{1}{1260},\ \delta_{24}\to-\frac{1}{2520},\ \delta_{25}\to\frac{1}{1680},$
$\delta_{26}\to\frac{1}{3360},\ \delta_{30}\to-\frac{19}{960},\ \delta_{31}\to\frac{1}{1920},\ \delta_{32}\to-\frac{11}{40320},\ \delta_{33}\to-\frac{1}{4480},\ \delta_{34}\to\frac{53}{100800},$
$\delta_{35}\to\frac{1}{5376},\ \delta_{40}\to-\frac{1}{80},\ \delta_{41}\to-\frac{53}{40320},\ \delta_{42}\to\frac{1}{8064},\ \delta_{43}\to\frac{167}{201600},\ \delta_{44}\to\frac{1}{9600},$
$\delta_{50}\to-\frac{17}{2016},\ \delta_{51}\to-\frac{5}{16128},\ \delta_{52}\to\frac{2347}{1935360},\ \delta_{53}\to\frac{37}{184320},\ \delta_{60}\to-\frac{3}{896},\ \delta_{61}\to\frac{97}{71680},$
$\delta_{62}\to\frac{7}{23040},\ \delta_{70}\to\frac{11}{30720},\ \delta_{71}\to\frac{7}{30720},\ \delta_{80}\to-\frac{1}{1152},\ \kappa_0\to 1,\ \kappa_2\to\frac{1}{48}\left(-1+48\kappa_1^2\right),$
$\kappa_3\to\frac{1}{16}\kappa_1\left(-1+16\kappa_1^2\right),\ \kappa_4\to\frac{13-480\kappa_1^2+3840\kappa_1^4}{3840},\ \kappa_5\to\frac{1}{768}\kappa_1\left(13-160\kappa_1^2+768\kappa_1^4\right),$
$\kappa_6\to\frac{-51+1456\kappa_1^2-8960\kappa_1^4+28672\kappa_1^6}{28672},\ \kappa_7\to\frac{\kappa_1\left(-153+1456\kappa_1^2-5376\kappa_1^4+12288\kappa_1^6\right)}{12288},$
$\kappa_8\to\frac{1169-29376\kappa_1^2+139776\kappa_1^4-344064\kappa_1^6+589824\kappa_1^8}{589824},\ \omega_0\to 1,\ \omega_1\to 0,\ \omega_2\to 0,\ \omega_3\to 0,$
$\omega_4\to 0,\ \omega_5\to 0,\ \omega_6\to 0,\ \omega_7\to 0,\ \omega_8\to 0,\ \omega_{10}\to 0,\ \omega_{11}\to-\frac{1}{48},\ \omega_{12}\to 0,\ \omega_{13}\to\frac{1}{480},\ \omega_{14}\to 0,$
$\omega_{15}\to-\frac{1}{1008},\ \omega_{16}\to 0,\ \omega_{17}\to\frac{1}{960},\ \omega_{20}\to 0,\ \omega_{21}\to 0,\ \omega_{22}\to\frac{17}{5760},\ \omega_{23}\to 0,\ \omega_{24}\to-\frac{3}{2240},$
$\omega_{25}\to 0,\ \omega_{26}\to\frac{37}{26880},\ \omega_{30}\to 0,\ \omega_{31}\to\frac{1}{480},\ \omega_{32}\to 0,\ \omega_{33}\to-\frac{103}{71680},\ \omega_{34}\to 0,\ \omega_{35}\to\frac{991}{645120},$

$\omega_{40} \to 0,\ \omega_{41} \to 0,\ \omega_{42} \to -\frac{3}{2240},\ \omega_{43} \to 0,\ \omega_{44} \to \frac{20509}{12902400},\ \omega_{50} \to 0,\ \omega_{51} \to -\frac{1}{1008},$
$\omega_{52} \to 0,\ \omega_{53} \to \frac{991}{645120},\ \omega_{60} \to 0,\ \omega_{61} \to 0,\ \omega_{62} \to \frac{37}{26880},\ \omega_{70} \to 0,\ \omega_{71} \to \frac{1}{960},\ \omega_{80} \to 0\}\}$

V1 == V2

True

C1 == (C2 /. κ_1 → 0)

True