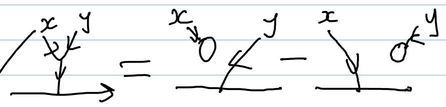
Idea. study $\mathcal{A}^{\beta}(\uparrow_X) := \mathcal{A}^{w}(\uparrow_X)\Big/$ 
using (renormalized) semi-symmetrized calculus. i.e. $[x,y] = c_x y - c_y x$

Proposition 1. The following diagram is commutative:

$$\begin{array}{c|c} w & hs \\ \hline x & \alpha \\ y & \beta \\ ts & \gamma \end{array} \xrightarrow[I^{rss}=\beta]{I^{ss}} \mathcal{A}^{\beta}(\uparrow_x \uparrow_y \uparrow_{ts} \uparrow_{hs})$$

$$\Big\downarrow tm^{xy}_z \qquad\qquad\qquad \Big\downarrow m^{xy}_z$$

$$\begin{array}{c|c} w & hs \\ \hline z & \alpha+\beta \\ ts & \gamma \end{array} \Big/ \begin{array}{l} c_x, c_y \\ \to c_z \end{array} \xrightarrow[I^{\beta}]{I^{ss}} \mathcal{A}^{\beta}(\uparrow_z \ldots)$$

Here $I^{ss}: \begin{array}{c|c} w & h_j \\ \hline t_i & \alpha_{ij} \end{array} \longmapsto \chi^{ss}\left(w \cdot \exp \sum \alpha_{ij} a_{ij}\right)$

and with $K_j = \log(1+\langle\alpha_j\rangle)/\langle\alpha_j\rangle$ (here $\langle\alpha\rangle := \sum c_i \alpha_i$),

$I^{\beta}: \begin{array}{c|c} w & h_j \\ \hline t_i & \alpha_{ij} \end{array} \longmapsto \chi^{ss}\left(w \exp \sum K_j \alpha_{ij} a_{ij}\right)$

Comment. We have

$$\begin{tikzcd} \beta(hs,ts) \arrow[rr, bend left, "K"] \arrow[dr, "I^{rss}=I^{\beta}"'] & & \beta(hs,ts) \arrow[ll, bend left, "J"] \arrow[dl, "I^{ss}"] \\ & \mathcal{A}^{\beta}(hs,ts) & \end{tikzcd}$$

with

$$\left(\begin{array}{c|c} w & \\ \hline & \alpha_{ij} \end{array}\right) \overset{K}{\longmapsto} \left(\begin{array}{c|c} w & \\ \hline & K_j\alpha_{ij} \end{array}\right)$$

$$\left(\begin{array}{c|c} w & \\ \hline & \end{array}\right) \overset{J}{\longmapsto} \left(\begin{array}{c|c} w & \\ \hline & J_j\alpha_{ij} \end{array}\right)$$

with K_j as above and with $J_j = (e^{\langle\alpha_j\rangle}-1)/\langle\alpha_j\rangle$.

Proposition 2. The following diagram is commutative:

$$\begin{array}{c|ccc} w & x & y & hs \\ \hline ts & \alpha & \beta & \gamma \end{array} \xrightarrow{I^{\beta}} \mathcal{A}^{\beta}(\uparrow_x \uparrow_y \cdots)$$

$$\Big\downarrow hm^{xy} \qquad\qquad\qquad \Big\downarrow m^{xy}$$

$$\begin{array}{ccc}
\cdot & & \\
\Big\downarrow{\scriptstyle hm^{xy}_z} & & \Big\downarrow{\scriptstyle m^{xy}_z} \\
\begin{array}{c|cc} w & z & hs \\ \hline ts & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array} & \xrightarrow{I^\beta} & \mathcal{A}^\beta(\uparrow_z \cdots)
\end{array}$$

Proposition 3. The following diagram is commutative:

$$\begin{array}{ccccc}
\begin{array}{c|cc} w & y & hs \\ \hline x & \alpha & \beta \\ ts & \gamma & \delta \end{array} & \xrightarrow{I^\beta} & \mathcal{A}^\beta(\uparrow_x\uparrow_y \cdots) & & \\
& & & \searrow{\scriptstyle m^{xy}_z} & \\
\Big\downarrow{\scriptstyle sw^{th}_{xy}} & & & & \mathcal{A}^\beta(\uparrow_z \cdots) \\
& & & \nearrow{\scriptstyle m^{yx}_z} & \\
\begin{array}{c|cc} w\cdot\epsilon & y & hs \\ \hline x & \alpha+\frac{\langle\gamma\rangle}{\epsilon}\alpha & \beta+\frac{\langle\gamma\rangle}{\epsilon}\beta \\ ts & \gamma/\epsilon & \delta-\frac{c_x}{\epsilon}\gamma\cdot\beta \end{array} & \xrightarrow{I^\beta} & \mathcal{A}^\beta(\uparrow_x\uparrow_y \cdots) & &
\end{array}$$

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thswap[x_, y_][B[ω_, μ_]] := Module[
   {α, β, γ, δ, ϵ},
   α = Coefficient[μ, h[y] t[x]];
   β = D[μ, t[x]] /. h[y] → 0;
   γ = D[μ, h[y]] /. t[x] → 0;
   δ = μ /. h[y] | t[x] → 0;
   ϵ = 1 + ℏ c_x α;
   B[ω ϵ, Plus[
      α (1 + (γ /. t[i_] ↦ ℏ c_i) / ϵ) h[y] t[x],
      β (1 + (γ /. t[i_] ↦ ℏ c_i) / ϵ) t[x],
      γ / ϵ h[y],
      δ - ℏ c_x / ϵ γ β
    ]] // βCollect
  ];
```

Proposition 4.

$$\begin{array}{ccc}
\left(\begin{array}{c|c} w_1 & hs_1 \\ \hline ts_1 & A_1 \end{array}, \begin{array}{c|c} w_2 & hs_2 \\ \hline ts_2 & A_2 \end{array}\right) & \xrightarrow[I^{ss}]{I^\beta} & \mathcal{A}^\beta(ts_1, hs_1) \otimes \mathcal{A}^\beta(ts_2, hs_2) \\
\Big\downarrow & & \Big\downarrow \\
\left(\begin{array}{c|cc} w_1w_2 & hs_1 & hs_2 \\ \hline ts_1 & A_1 & 0 \\ ts_2 & 0 & A_2 \end{array}\right) & \xrightarrow[I^{ss}]{I^\beta} & \mathcal{A}^\beta(ts_1 \cup ts_2, hs_1 \cup hs_2)
\end{array}$$

Propsition 0. $R_{xy} := \begin{array}{c|c} 1 & y \\ \hline x & \frac{e^{c_x}-1}{c_x} \end{array} \xmapsto{I^\beta} R_{xy} = \uparrow_x \xrightarrow{e^a} \uparrow_y$

and $R^{-1}_{xy} := \begin{array}{c|c} 1 & y \\ \hline x & \frac{e^{-c_x}-1}{c_x} \end{array} \xmapsto{I^\beta} R^{-1}_{xy} = \uparrow_x \xrightarrow{e^{-1}} \uparrow_y$.