Deriving the controlled-sum formulas

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From http://www.math.toronto.edu/~drorbn/Talks/GWU-1203:

$$tm^{xy}_z : \begin{array}{c|c} \omega & \cdots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

here
$\langle\alpha\rangle = \sigma_x - 1$

$$sw^{th}_{xy} : \begin{array}{c|cc} \omega & h_y & \cdots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \cdots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

$\langle\gamma\rangle = \sigma_y - 1 - \alpha$

where $\epsilon := 1+\alpha$, $\langle\alpha\rangle := \sum_i \alpha_i$, and $\langle\gamma\rangle := \sum_{i\neq x}\gamma_i$, and let

Let $\alpha' = W\alpha$, $\beta' = W\beta$ etc. Then

hm:

β' world $\left\{ \begin{array}{c|ccc} W & x & y & - \\ \hline & \alpha' & \beta' & \gamma' \end{array} \right.$ $\qquad\qquad \begin{array}{c|cc} W & z & - \\ \hline & \alpha+\sigma_x\beta & \gamma \end{array}$

$\downarrow \qquad\qquad\qquad\qquad \uparrow$

β world $\left\{ \begin{array}{c|ccc} W & x & y & - \\ \hline & \alpha'/W & \beta'/W & \gamma'/W \end{array} \right. \longrightarrow \begin{array}{c|cc} W & z & - \\ \hline & \frac{\alpha'}{W}+\frac{\beta'}{W}+\langle\frac{\alpha'}{W}\rangle\frac{\beta'}{W} & \frac{\gamma'}{W} \end{array}$

Recall that $\langle\alpha\rangle = \sigma_x - 1$ & $\langle\alpha'\rangle = W\sigma_x - W$.

$\begin{array}{c|cc} W & y & - \\ \hline x & \alpha' & \beta' \\ 1 & \gamma' & \delta' \end{array}$ in β'-world

$\downarrow$ β' to β

Asides:

$1 + \underline{W\sigma_y - W - \alpha'} = \underline{W\sigma_y}$

$\big\downarrow$ β' to β

$$\begin{array}{c|cc} w & y & - \\ \hline x & \alpha'/w & \beta'/w \\ 1 & \gamma'/w & \delta'/w \end{array}$$

$\big\downarrow$ sw^{th}_{xy}

$$\begin{array}{c|c|c} w\left(1+\frac{\alpha'}{w}\right) & y & - \\ \hline x & \multicolumn{2}{c}{\left[1+\left(\sigma_y - 1 - \frac{\alpha'}{w}\right)\Big/\left(1+\frac{\alpha'}{w}\right)\right]\cdot\left(\frac{\alpha'}{w} \quad \frac{\beta'}{w}\right)} \\ \hline 1 & \frac{\gamma'}{w}\Big/\left(1+\frac{\alpha'}{w}\right) & \frac{\delta'}{w} - \frac{\gamma'\beta'}{w^2}\Big/\left(1+\frac{\alpha'}{w}\right) \end{array}$$

$$= \begin{array}{c|c|c} w+\alpha' & y & - \\ \hline x & \multicolumn{2}{c}{\left(\frac{w\sigma_y}{w+\alpha'}\right)\left(\frac{\alpha'}{w} \quad \frac{\beta'}{w}\right)} \\ \hline 1 & \frac{\gamma'}{w+\alpha'} & \frac{\delta'}{w} - \frac{\gamma'\beta'}{w}\Big/(w+\alpha') \end{array}$$

$\big\downarrow$ β to β'

$$\begin{array}{c|c|c} w+\alpha' & y & - \\ \hline x & \multicolumn{2}{c}{\sigma_y\left(\alpha' \quad \beta'\right)} \\ \hline 1 & \gamma' & \left[(w+\alpha')\delta' - \gamma'\beta'\right]/w \end{array}$$

$$1+\frac{w\sigma_y - w - \alpha'}{w+\alpha'} = \frac{w\sigma_y}{w+\alpha'}$$