An Euler proof of BCH

April-17-12
7:04 PM

See 2008-08/ D, exp, log

$$e^x e^y = e^{bch(x,y)} = e^\phi$$

<u>Claim</u> $\phi = bch(x,y) \in Lie(x,y)$

<u>proof</u> Apply $\tilde{E}$: $\qquad [\tilde{E}Z = Z^{-1}EZ]$

$$lhs \to e^{-y}e^{-x}(xe^xe^y + e^xye^y) = e^{-y}xe^y + y$$

$\in Lie(x,y)$

In general, $E(e^D) = e^D \frac{1-e^{-ad\, D}}{ad\, D}(ED)$

Aside $e^{A+\epsilon B} = e^A\left(1+\epsilon\frac{1-e^{-ad\, A}}{ad\, A}(B)\right) + o(\epsilon^2)$

i.e., $de^A = e^A\frac{1-e^{-ad\, A}}{ad\, A}$

So $\frac{1-e^{-ad\,\phi}}{ad\,\phi}(E\phi) = e^{-y}xe^y + y \in Lie(x,y)$

<u>claim</u> ϕ exists and is unique and belongs to $Lie(x,y)$.

$$E\phi - \frac{1}{2}[\phi, E\phi] + \frac{1}{6}[\phi,[\phi,E\phi]] - \dots = e^{-ad\, y}x + y$$