Pensieve header: Perturbative β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-04"];
<< betaCalculus.m
```

```
Clear[ħ];
$PerturbativeDegree = 4;
βSimplify[expr_] := Replace[
   Series[Normal[expr], {ħ, 0, $PerturbativeDegree}],
   sd_SeriesData :> MapAt[Expand, sd, 3]
  ];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   βSimplify[μ]
  ];
```

The Knot-Theoretic Equations

```
{
  V0 = βCollect[
     B[ω[ħ c1, ħ c2], α[ħ c1, ħ c2] t[1] h[1] +
       β[ħ c1, ħ c2] t[1] h[2] + γ[ħ c1, ħ c2] t[2] h[1] + δ[ħ c1, ħ c2] t[2] h[2]]
    ] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  C0 = βCollect[B[κ[ħ c1], 0]] /. {
     (ϵ : (α | β | γ | δ | ω | κ))[___] :> ϵ0, (ϵ : (α | β | γ | δ | ω | κ))^(k___)[___] :> ϵFromDigits[{k}]
    },
  eqns1 = HardR4[V0],
  eqns2 = TwistEq[V0],
  eqns3 = And[(V0 // dη[1]) == B[1, 0], (V0 // dη[2]) == B[1, 0]],
  eqns4 = V0 ** (V0 // dA[1] // dA[2]) == B[1, 0],
  eqns5 = CapEquation[V0, C0],
  eqns6 = (C0 // tη[1]) == B[1, 0],
  eqns7 = (V0 == Rot120[V0]),
  eqns8 = V0 ** (V0 // dS[1] // dS[2]) == R[2, 1]
 } // ColumnForm
```

A very large output was generated. Here is a sample of it:

<<1>>

Show Less | Show More | Show Full Output | Set Size Limit...

```
eqns = eqns1 && eqns2 && eqns3 && eqns4 && eqns5 && eqns6 && eqns8;
```

```
sol = SolveAlways[eqns, {ħ, c1, c2}]
```

$$\left\{\left\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\left(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}\right),\ \beta_4 \to 0,\right.\right.$$

$$\gamma_4 \to \frac{1}{480}\,(1 - 240\,\beta_1 + 960\,\gamma_{30}),\ \alpha_{13} \to \frac{1}{640}\,(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\,(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\,(1 + 40\,\beta_1 - 80\,\gamma_{30}),$$

$$\gamma_{31} \to \frac{1}{128}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),\ \gamma_{40} \to \frac{1}{40}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,$$

$$\omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),\ \beta_{21} \to \frac{1}{192}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12}),$$

$$\beta_{12} \to \frac{1}{180}\,(-7 - 150\,\beta_1 + 180\,\gamma_{12}),\ \gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},$$

$$\beta_3 \to \frac{1}{64}\,(-1 - 48\,\beta_1 + 64\,\gamma_{30}),\ \beta_{30} \to \frac{1}{80}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),\ \alpha_{30} \to 0,$$

$$\alpha_3 \to \frac{1}{64}\,(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}),\ \alpha_{12} \to \frac{1}{192}\,(-5 - 144\,\beta_1 + 192\,\delta_{21}),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\,(11 + 80\,\beta_1 + 320\,\gamma_{30}),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$

$$\alpha_2 \to \frac{1}{8}\,(-1 - 8\,\beta_1 + 8\,\delta_{10}),\ \alpha_{11} \to \frac{1}{48}\,(-1 - 24\,\beta_1),\ \delta_2 \to 0,\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$

$$\gamma_{11} \to \frac{1}{16}\,(1 + 16\,\beta_1),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\,(1 + 12\,\beta_1),\ \gamma_{20} \to \frac{1}{24}\,(1 + 24\,\beta_1),$$

$$\omega_{11} \to \frac{1}{16}\,(-1 + 16\,\delta_{10}),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),$$

$$\gamma_1 \to \frac{1}{6}\,(1 + 6\,\beta_1),\ \beta_{10} \to \frac{1}{24}\,(1 + 24\,\beta_1),\ \gamma_{10} \to \frac{1}{8}\,(1 + 8\,\beta_1),\ \alpha_{10} \to 0,\ \alpha_1 \to \frac{1}{8}\,(-1 + 8\,\delta_{10}),$$

$$\delta_1 \to 0,\ \omega_1 \to 0,\ \omega_{10} \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\,(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\,(1 - 240\,\beta_1 + 960\,\gamma_{30}),\ \alpha_{13} \to \frac{1}{640}\,(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\,(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\,(1 + 40\,\beta_1 - 80\,\gamma_{30}),$$

$$\gamma_{31} \to \frac{1}{128}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),\ \gamma_{40} \to \frac{1}{40}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12}),\ \beta_{12} \to \frac{1}{180}\,(-7 - 150\,\beta_1 + 180\,\gamma_{12}),$$

$$\gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\,(-1 - 48\,\beta_1 + 64\,\gamma_{30}),\ \beta_{30} \to \frac{1}{80}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\,(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}),\ \alpha_{12} \to \frac{1}{192}\,(-5 - 144\,\beta_1 + 192\,\delta_{21}),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\,(11 + 80\,\beta_1 + 320\,\gamma_{30}),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),\ \omega_{12} \to 0,$$

$$\omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,\ \alpha_2 \to \frac{1}{8}\,(-1 - 8\,\beta_1 + 8\,\delta_{10}),$$

$$\alpha_{11} \to \frac{1}{48}\,(-1 - 24\,\beta_1),\ \delta_2 \to 0,\ \delta_{11} \to 0,\ \delta_{20} \to 0,\ \gamma_{11} \to \frac{1}{16}\,(1 + 16\,\beta_1),\ \beta_{20} \to 0,\ \beta_{11} \to 0,$$

$$\beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\,(1 + 12\,\beta_1),\ \gamma_{20} \to \frac{1}{24}\,(1 + 24\,\beta_1),\ \omega_{11} \to \frac{1}{16}\,(-1 + 16\,\delta_{10}),\ \omega_2 \to 0,\ \omega_{20} \to 0,$$

$$\kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),\ \gamma_1 \to \frac{1}{6}\,(1 + 6\,\beta_1),\ \beta_{10} \to \frac{1}{24}\,(1 + 24\,\beta_1),\ \gamma_{10} \to \frac{1}{8}\,(1 + 8\,\beta_1),\ \alpha_{10} \to 0,$$

$$\alpha_1 \to \frac{1}{8}\,(-1 + 8\,\delta_{10}),\ \delta_1 \to 0,\ \omega_1 \to 0,\ \omega_{10} \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\,(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\,(1 - 240\,\beta_1 + 960\,\gamma_{30}),\ \alpha_{13} \to \frac{1}{640}\,(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\,(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\,(1 + 40\,\beta_1 - 80\,\gamma_{30}),$$

$$\gamma_{31} \to \frac{1}{128}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),\ \gamma_{40} \to \frac{1}{40}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\,(-7 - 144\,\beta_1 + 192\,\gamma_{12}),\ \beta_{12} \to \frac{1}{180}\,(-7 - 150\,\beta_1 + 180\,\gamma_{12}),$$

$$\gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\,(-1 - 48\,\beta_1 + 64\,\gamma_{30}),\ \beta_{30} \to \frac{1}{80}\,(-1 - 40\,\beta_1 + 80\,\gamma_{30}),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\,(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}),\ \alpha_{12} \to \frac{1}{192}\,(-5 - 144\,\beta_1 + 192\,\delta_{21}),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\,(11 + 80\,\beta_1 + 320\,\gamma_{30}),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),\ \omega_{12} \to 0,$$

$$\omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,\ \alpha_2 \to \frac{1}{8}\,(-1 - 8\,\beta_1 + 8\,\delta_{10}),$$

$$\alpha_{11} \to \frac{1}{48}\,(-1 - 24\,\beta_1),\ \delta_2 \to 0,\ \delta_{11} \to 0,\ \delta_{20} \to 0,\ \gamma_{11} \to \frac{1}{16}\,(1 + 16\,\beta_1),\ \beta_{20} \to 0,\ \beta_{11} \to 0,$$

$$\beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\,(1 + 12\,\beta_1),\ \gamma_{20} \to \frac{1}{24}\,(1 + 24\,\beta_1),\ \omega_{11} \to \frac{1}{16}\,(-1 + 16\,\delta_{10}),\ \omega_2 \to 0,\ \omega_{20} \to 0,$$

$$\kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),\ \gamma_1 \to \frac{1}{6}\,(1 + 6\,\beta_1),\ \gamma_{10} \to \frac{1}{8}\,(1 + 8\,\beta_1),\ \beta_{10} \to \frac{1}{24}\,(1 + 24\,\beta_1),\ \alpha_{10} \to 0,$$

$$\alpha_1 \to \frac{1}{8}\,(-1 + 8\,\delta_{10}),\ \delta_1 \to 0,\ \omega_1 \to 0,\ \omega_{10} \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\,(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\,(1 - 240\,\beta_1 + 960\,\gamma_{30}),\ \alpha_{13} \to \frac{1}{640}\,(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\,(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\,(1 + 40\,\beta_1 - 80\,\gamma_{30}),$$

$$\gamma_{31} \to \frac{1}{128}\,(-7-144\,\beta_1+192\,\gamma_{12}+64\,\gamma_{30}),\ \gamma_{40} \to \frac{1}{40}\,(-1-40\,\beta_1+80\,\gamma_{30}),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13+192\,\beta_1+4608\,\beta_1^2-288\,\delta_{10}-4608\,\beta_1\,\delta_{10}+2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\,\delta_{10}^2+576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\,\delta_{10}^2+576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35+384\,\beta_1+9216\,\beta_1^2-864\,\delta_{10}-9216\,\beta_1\,\delta_{10}+2304\,\delta_{10}^2+4608\,\delta_{21}-864\,\kappa_1^2+13\,824\,\delta_{10}\,\kappa_1^2+2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\,(-7-144\,\beta_1+192\,\gamma_{12}),\ \beta_{12} \to \frac{1}{180}\,(-7-150\,\beta_1+180\,\gamma_{12}),$$

$$\gamma_{21} \to \frac{-37-240\,\beta_1+2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\,(-1-48\,\beta_1+64\,\gamma_{30}),\ \beta_{30} \to \frac{1}{80}\,(-1-40\,\beta_1+80\,\gamma_{30}),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\,(-7-96\,\beta_1+48\,\delta_{10}+64\,\delta_{30}),\ \alpha_{12} \to \frac{1}{192}\,(-5-144\,\beta_1+192\,\delta_{21}),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7-192\,\beta_1+576\,\beta_1^2+72\,\delta_{10}-576\,\beta_1\,\delta_{10}-1152\,\delta_{10}^2+1152\,\delta_{21}-288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\,(11+80\,\beta_1+320\,\gamma_{30}),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5+48\,\beta_1+1152\,\beta_1^2+144\,\delta_{10}-1152\,\beta_1\,\delta_{10}-2304\,\delta_{10}^2+2304\,\delta_{21}-576\,\delta_{30}\right),$$

$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3+48\,\delta_{10}+16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$

$$\alpha_2 \to \frac{1}{8}\,(-1-8\,\beta_1+8\,\delta_{10}),\ \alpha_{11} \to \frac{1}{48}\,(-1-24\,\beta_1),\ \delta_2 \to 0,\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$

$$\gamma_{11} \to \frac{1}{16}\,(1+16\,\beta_1),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\,(1+12\,\beta_1),\ \gamma_{20} \to \frac{1}{24}\,(1+24\,\beta_1),$$

$$\omega_{11} \to \frac{1}{16}\,(-1+16\,\delta_{10}),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1+16\,\delta_{10}+16\,\kappa_1^2\right),$$

$$\gamma_1 \to \frac{1}{6}\,(1+6\,\beta_1),\ \gamma_{10} \to \frac{1}{8}\,(1+8\,\beta_1),\ \beta_{10} \to \frac{1}{24}\,(1+24\,\beta_1),\ \alpha_{10} \to 0,\ \alpha_1 \to \frac{1}{8}\,(-1+8\,\delta_{10}),$$

$$\delta_1 \to 0,\ \omega_1 \to 0,\ \omega_{10} \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\,(-1-8\,\beta_1-32\,\gamma_{30}+8\,\delta_{10}+32\,\delta_{30}),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\,(1-240\,\beta_1+960\,\gamma_{30}),\ \alpha_{13} \to \frac{1}{640}\,(19+240\,\beta_1-960\,\gamma_{12}+960\,\delta_{21}),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\,(-5-112\,\beta_1+192\,\gamma_{12}+64\,\gamma_{30}),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49+912\,\beta_1+1152\,\beta_1^2-1728\,\gamma_{12}+144\,\delta_{10}-1152\,\beta_1\,\delta_{10}-2304\,\delta_{10}^2+2304\,\delta_{21}-576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169-3120\,\beta_1+5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\,(1+40\,\beta_1-80\,\gamma_{30}),$$

$$\gamma_{31} \to \frac{1}{128}\,(-7-144\,\beta_1+192\,\gamma_{12}+64\,\gamma_{30}),\ \gamma_{40} \to \frac{1}{40}\,(-1-40\,\beta_1+80\,\gamma_{30}),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12}\right),\ \beta_{12} \to \frac{1}{180}\left(-7 - 150\,\beta_1 + 180\,\gamma_{12}\right),$$

$$\gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\left(-1 - 48\,\beta_1 + 64\,\gamma_{30}\right),\ \beta_{30} \to \frac{1}{80}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\left(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}\right),\ \alpha_{12} \to \frac{1}{192}\left(-5 - 144\,\beta_1 + 192\,\delta_{21}\right),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\left(11 + 80\,\beta_1 + 320\,\gamma_{30}\right),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$

$$\delta_2 \to 0,\ \alpha_2 \to \frac{1}{8}\left(-1 - 8\,\beta_1 + 8\,\delta_{10}\right),\ \alpha_{11} \to \frac{1}{48}\left(-1 - 24\,\beta_1\right),\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$

$$\gamma_{11} \to \frac{1}{16}\left(1 + 16\,\beta_1\right),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\left(1 + 12\,\beta_1\right),\ \gamma_{20} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),$$

$$\omega_{11} \to \frac{1}{16}\left(-1 + 16\,\delta_{10}\right),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),$$

$$\gamma_1 \to \frac{1}{6}\left(1 + 6\,\beta_1\right),\ \beta_{10} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),\ \gamma_{10} \to \frac{1}{8}\left(1 + 8\,\beta_1\right),\ \alpha_{10} \to 0,\ \delta_1 \to 0,$$

$$\alpha_1 \to \frac{1}{8}\left(-1 + 8\,\delta_{10}\right),\ \omega_{10} \to 0,\ \omega_1 \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\left(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}\right),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\left(1 - 240\,\beta_1 + 960\,\gamma_{30}\right),\ \alpha_{13} \to \frac{1}{640}\left(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}\right),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\left(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\left(1 + 40\,\beta_1 - 80\,\gamma_{30}\right),$$

$$\gamma_{31} \to \frac{1}{128}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),\ \gamma_{40} \to \frac{1}{40}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12}\right),\ \beta_{12} \to \frac{1}{180}\left(-7 - 150\,\beta_1 + 180\,\gamma_{12}\right),$$

$$\gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\left(-1 - 48\,\beta_1 + 64\,\gamma_{30}\right),\ \beta_{30} \to \frac{1}{80}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\left(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}\right),\ \alpha_{12} \to \frac{1}{192}\left(-5 - 144\,\beta_1 + 192\,\delta_{21}\right),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\left(11 + 80\,\beta_1 + 320\,\gamma_{30}\right),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$

$$\delta_2 \to 0,\ \alpha_2 \to \frac{1}{8}\left(-1 - 8\,\beta_1 + 8\,\delta_{10}\right),\ \alpha_{11} \to \frac{1}{48}\left(-1 - 24\,\beta_1\right),\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$

$$\gamma_{11} \to \frac{1}{16}\left(1 + 16\,\beta_1\right),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\left(1 + 12\,\beta_1\right),\ \gamma_{20} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),$$

$$\omega_{11} \to \frac{1}{16}\left(-1 + 16\,\delta_{10}\right),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),$$

$$\gamma_1 \to \frac{1}{6}\left(1 + 6\,\beta_1\right),\ \beta_{10} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),\ \gamma_{10} \to \frac{1}{8}\left(1 + 8\,\beta_1\right),\ \alpha_{10} \to 0,\ \delta_1 \to 0,$$

$$\alpha_1 \to \frac{1}{8}\left(-1 + 8\,\delta_{10}\right),\ \omega_{10} \to 0,\ \omega_1 \to 0,\ \kappa_0 \to 1,\ \delta_0 \to 0,\ \alpha_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\left(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}\right),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\left(1 - 240\,\beta_1 + 960\,\gamma_{30}\right),\ \alpha_{13} \to \frac{1}{640}\left(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}\right),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\left(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\left(1 + 40\,\beta_1 - 80\,\gamma_{30}\right),$$

$$\gamma_{31} \to \frac{1}{128}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),\ \gamma_{40} \to \frac{1}{40}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12}\right),\ \beta_{12} \to \frac{1}{180}\left(-7 - 150\,\beta_1 + 180\,\gamma_{12}\right),$$

$$\gamma_{21} \to \frac{-37 - 240\,\beta_1 + 2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\left(-1 - 48\,\beta_1 + 64\,\gamma_{30}\right),\ \beta_{30} \to \frac{1}{80}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),$$

$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\left(-7 - 96\,\beta_1 + 48\,\delta_{10} + 64\,\delta_{30}\right),\ \alpha_{12} \to \frac{1}{192}\left(-5 - 144\,\beta_1 + 192\,\delta_{21}\right),$$

$$\alpha_{21} \to \frac{1}{864}\left(-7 - 192\,\beta_1 + 576\,\beta_1^2 + 72\,\delta_{10} - 576\,\beta_1\,\delta_{10} - 1152\,\delta_{10}^2 + 1152\,\delta_{21} - 288\,\delta_{30}\right),$$

$$\gamma_3 \to \frac{1}{320}\left(11 + 80\,\beta_1 + 320\,\gamma_{30}\right),\ \delta_3 \to 0,$$

$$\delta_{12} \to \frac{1}{1728}\left(-5 + 48\,\beta_1 + 1152\,\beta_1^2 + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3 + 48\,\delta_{10} + 16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$

$$\delta_2 \to 0,\ \alpha_2 \to \frac{1}{8}\left(-1 - 8\,\beta_1 + 8\,\delta_{10}\right),\ \alpha_{11} \to \frac{1}{48}\left(-1 - 24\,\beta_1\right),\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$

$$\gamma_{11} \to \frac{1}{16}\left(1 + 16\,\beta_1\right),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\left(1 + 12\,\beta_1\right),\ \gamma_{20} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),$$

$$\omega_{11} \to \frac{1}{16}\left(-1 + 16\,\delta_{10}\right),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1 + 16\,\delta_{10} + 16\,\kappa_1^2\right),$$

$$\gamma_1 \to \frac{1}{6}\left(1 + 6\,\beta_1\right),\ \gamma_{10} \to \frac{1}{8}\left(1 + 8\,\beta_1\right),\ \beta_{10} \to \frac{1}{24}\left(1 + 24\,\beta_1\right),\ \alpha_{10} \to 0,\ \delta_1 \to 0,$$

$$\alpha_1 \to \frac{1}{8}\left(-1 + 8\,\delta_{10}\right),\ \omega_{10} \to 0,\ \omega_1 \to 0,\ \kappa_0 \to 1,\ \alpha_0 \to 0,\ \delta_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\},$$

$$\Big\{\alpha_{40} \to 0,\ \beta_{40} \to 0,\ \alpha_4 \to \frac{1}{16}\left(-1 - 8\,\beta_1 - 32\,\gamma_{30} + 8\,\delta_{10} + 32\,\delta_{30}\right),\ \beta_4 \to 0,$$

$$\gamma_4 \to \frac{1}{480}\left(1 - 240\,\beta_1 + 960\,\gamma_{30}\right),\ \alpha_{13} \to \frac{1}{640}\left(19 + 240\,\beta_1 - 960\,\gamma_{12} + 960\,\delta_{21}\right),$$

$$\beta_{13} \to 0,\ \beta_{31} \to 0,\ \gamma_{13} \to \frac{1}{128}\left(-5 - 112\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),$$

$$\alpha_{22} \to \frac{1}{1728}\left(49 + 912\,\beta_1 + 1152\,\beta_1^2 - 1728\,\gamma_{12} + 144\,\delta_{10} - 1152\,\beta_1\,\delta_{10} - 2304\,\delta_{10}^2 + 2304\,\delta_{21} - 576\,\delta_{30}\right),$$

$$\beta_{22} \to 0,\ \gamma_{22} \to \frac{-169 - 3120\,\beta_1 + 5760\,\gamma_{12}}{2880},\ \alpha_{31} \to \frac{1}{160}\left(1 + 40\,\beta_1 - 80\,\gamma_{30}\right),$$

$$\gamma_{31} \to \frac{1}{128}\left(-7 - 144\,\beta_1 + 192\,\gamma_{12} + 64\,\gamma_{30}\right),\ \gamma_{40} \to \frac{1}{40}\left(-1 - 40\,\beta_1 + 80\,\gamma_{30}\right),\ \delta_4 \to 0,\ \delta_{13} \to 0,$$

$$\delta_{22} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \omega_{22} \to \frac{1}{1152}\left(13 + 192\,\beta_1 + 4608\,\beta_1^2 - 288\,\delta_{10} - 4608\,\beta_1\,\delta_{10} + 2304\,\delta_{21}\right),$$

$$\omega_{31} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),$$

$$\omega_{13} \to \frac{1}{288}\left(1 + 48\,\beta_1 + 1152\,\beta_1^2 - 1152\,\beta_1\,\delta_{10} - 576\,\delta_{10}^2 + 576\,\delta_{21}\right),\ \omega_4 \to 0,\ \omega_{40} \to 0,\ \kappa_4 \to \frac{1}{2304}\left(35 + 384\,\beta_1 + 9216\,\beta_1^2 - 864\,\delta_{10} - 9216\,\beta_1\,\delta_{10} + 2304\,\delta_{10}^2 + 4608\,\delta_{21} - 864\,\kappa_1^2 + 13\,824\,\delta_{10}\,\kappa_1^2 + 2304\,\kappa_1^4\right),$$

$$\beta_{21} \to \frac{1}{192}\,(-7-144\,\beta_1+192\,\gamma_{12}),\ \beta_{12} \to \frac{1}{180}\,(-7-150\,\beta_1+180\,\gamma_{12}),$$
$$\gamma_{21} \to \frac{-37-240\,\beta_1+2880\,\gamma_{12}}{2880},\ \beta_3 \to \frac{1}{64}\,(-1-48\,\beta_1+64\,\gamma_{30}),\ \beta_{30} \to \frac{1}{80}\,(-1-40\,\beta_1+80\,\gamma_{30}),$$
$$\alpha_{30} \to 0,\ \alpha_3 \to \frac{1}{64}\,(-7-96\,\beta_1+48\,\delta_{10}+64\,\delta_{30}),\ \alpha_{12} \to \frac{1}{192}\,(-5-144\,\beta_1+192\,\delta_{21}),$$
$$\alpha_{21} \to \frac{1}{864}\left(-7-192\,\beta_1+576\,\beta_1^2+72\,\delta_{10}-576\,\beta_1\,\delta_{10}-1152\,\delta_{10}^2+1152\,\delta_{21}-288\,\delta_{30}\right),$$
$$\gamma_3 \to \frac{1}{320}\,(11+80\,\beta_1+320\,\gamma_{30}),\ \delta_3 \to 0,$$
$$\delta_{12} \to \frac{1}{1728}\left(-5+48\,\beta_1+1152\,\beta_1^2+144\,\delta_{10}-1152\,\beta_1\,\delta_{10}-2304\,\delta_{10}^2+2304\,\delta_{21}-576\,\delta_{30}\right),$$
$$\omega_{12} \to 0,\ \omega_{21} \to 0,\ \omega_{30} \to 0,\ \omega_3 \to 0,\ \kappa_3 \to \frac{1}{16}\,\kappa_1\left(-3+48\,\delta_{10}+16\,\kappa_1^2\right),\ \alpha_{20} \to 0,$$
$$\delta_2 \to 0,\ \alpha_2 \to \frac{1}{8}\,(-1-8\,\beta_1+8\,\delta_{10}),\ \alpha_{11} \to \frac{1}{48}\,(-1-24\,\beta_1),\ \delta_{11} \to 0,\ \delta_{20} \to 0,$$
$$\gamma_{11} \to \frac{1}{16}\,(1+16\,\beta_1),\ \beta_{20} \to 0,\ \beta_{11} \to 0,\ \beta_2 \to 0,\ \gamma_2 \to \frac{1}{12}\,(1+12\,\beta_1),\ \gamma_{20} \to \frac{1}{24}\,(1+24\,\beta_1),$$
$$\omega_{11} \to \frac{1}{16}\,(-1+16\,\delta_{10}),\ \omega_2 \to 0,\ \omega_{20} \to 0,\ \kappa_2 \to \frac{1}{16}\left(-1+16\,\delta_{10}+16\,\kappa_1^2\right),$$
$$\gamma_1 \to \frac{1}{6}\,(1+6\,\beta_1),\ \gamma_{10} \to \frac{1}{8}\,(1+8\,\beta_1),\ \beta_{10} \to \frac{1}{24}\,(1+24\,\beta_1),\ \alpha_{10} \to 0,\ \delta_1 \to 0,$$
$$\alpha_1 \to \frac{1}{8}\,(-1+8\,\delta_{10}),\ \omega_{10} \to 0,\ \omega_1 \to 0,\ \kappa_0 \to 1,\ \delta_0 \to 0,\ \alpha_0 \to 0,\ \beta_0 \to 0,\ \gamma_0 \to \frac{1}{2},\ \omega_0 \to 1\Big\}\Big\}$$

```
Length /@ {sol, sol = Union[Union /@ sol]}
```

$\{8, 1\}$

```
{V0, C0} /. sol[[1]]
```

$$\left\{\begin{pmatrix} 1+\frac{1}{16}\,c_1\,c_2\,(-1+16\,\delta_{10})\,\hbar^2+\left(\frac{c_1^3\,c_2\,\left(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\,\delta_{10}^2+576\,\delta_{21}\right)}{1728}+\frac{c_1\,c_2^3\,\left(1+48\,\beta_1+1152\,\beta_1^2-1152\,\beta_1\,\delta_{10}-576\right.}{1728}\right. \\ t[1] \\ t[2] \end{pmatrix}\right.$$

```
(V0 /. sol[[1]]) // Rot120 // dη[1] // dP[2 → 1]
```

$$\begin{pmatrix} 1+\frac{1}{48}\,c_1^2\,\hbar^2+\frac{c_1^4\,\hbar^4}{23\,040}+O[\hbar]^5 & h[1] \\ t[1] & -\frac{1}{2}+\frac{c_1\,\hbar}{8}-\frac{1}{48}\,c_1^2\,\hbar^2+\frac{1}{384}\,c_1^3\,\hbar^3-\frac{c_1^4\,\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

```
(V0 /. sol[[1]]) // Rot120 // dη[1] // dP[2 → 1] // dA[1] // dcap[1]
```

$$\begin{pmatrix} 1+\frac{1}{48}\,c_1^2\,\hbar^2+\frac{c_1^4\,\hbar^4}{23\,040}+O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
indvars = Union[Flatten[Union[Cases[Last /@ #, ϵ_k_ :> ϵ_k, Infinity]] & /@ sol]]
```

$\{\beta_1, \gamma_{12}, \gamma_{30}, \delta_{10}, \delta_{21}, \delta_{30}, \kappa_1\}$

```
sol1 = Union[
  sol[[1]] /. Thread[indvars → 0],
  Thread[indvars → 0]
 ]
```

$$\left\{\alpha_0 \to 0,\ \alpha_1 \to -\frac{1}{8},\ \alpha_2 \to -\frac{1}{8},\ \alpha_3 \to -\frac{7}{64},\ \alpha_4 \to -\frac{1}{16},\ \alpha_{10} \to 0,\ \alpha_{11} \to -\frac{1}{48},\ \alpha_{12} \to -\frac{5}{192},\right.$$
$$\alpha_{13} \to \frac{19}{640},\ \alpha_{20} \to 0,\ \alpha_{21} \to -\frac{7}{864},\ \alpha_{22} \to \frac{49}{1728},\ \alpha_{30} \to 0,\ \alpha_{31} \to \frac{1}{160},\ \alpha_{40} \to 0,\ \beta_0 \to 0,$$
$$\beta_1 \to 0,\ \beta_2 \to 0,\ \beta_3 \to -\frac{1}{64},\ \beta_4 \to 0,\ \beta_{10} \to \frac{1}{24},\ \beta_{11} \to 0,\ \beta_{12} \to -\frac{7}{180},\ \beta_{13} \to 0,\ \beta_{20} \to 0,$$
$$\beta_{21} \to -\frac{7}{192},\ \beta_{22} \to 0,\ \beta_{30} \to -\frac{1}{80},\ \beta_{31} \to 0,\ \beta_{40} \to 0,\ \gamma_0 \to \frac{1}{2},\ \gamma_1 \to \frac{1}{6},\ \gamma_2 \to \frac{1}{12},\ \gamma_3 \to \frac{11}{320},$$
$$\gamma_4 \to \frac{1}{480},\ \gamma_{10} \to \frac{1}{8},\ \gamma_{11} \to \frac{1}{16},\ \gamma_{12} \to 0,\ \gamma_{13} \to -\frac{5}{128},\ \gamma_{20} \to \frac{1}{24},\ \gamma_{21} \to -\frac{37}{2880},\ \gamma_{22} \to -\frac{169}{2880},$$
$$\gamma_{30} \to 0,\ \gamma_{31} \to -\frac{7}{128},\ \gamma_{40} \to -\frac{1}{40},\ \delta_0 \to 0,\ \delta_1 \to 0,\ \delta_2 \to 0,\ \delta_3 \to 0,\ \delta_4 \to 0,\ \delta_{10} \to 0,\ \delta_{11} \to 0,$$
$$\delta_{12} \to -\frac{5}{1728},\ \delta_{13} \to 0,\ \delta_{20} \to 0,\ \delta_{21} \to 0,\ \delta_{22} \to 0,\ \delta_{30} \to 0,\ \delta_{31} \to 0,\ \delta_{40} \to 0,\ \kappa_0 \to 1,$$
$$\kappa_1 \to 0,\ \kappa_2 \to -\frac{1}{16},\ \kappa_3 \to 0,\ \kappa_4 \to \frac{35}{2304},\ \omega_0 \to 1,\ \omega_1 \to 0,\ \omega_2 \to 0,\ \omega_3 \to 0,\ \omega_4 \to 0,\ \omega_{10} \to 0,$$
$$\left.\omega_{11} \to -\frac{1}{16},\ \omega_{12} \to 0,\ \omega_{13} \to \frac{1}{288},\ \omega_{20} \to 0,\ \omega_{21} \to 0,\ \omega_{22} \to \frac{13}{1152},\ \omega_{30} \to 0,\ \omega_{31} \to \frac{1}{288},\ \omega_{40} \to 0\right\}$$

```
V1 = V0 /. sol1
```

$$\begin{pmatrix} 1 - \frac{1}{16}\,(c_1\,c_2)\,\hbar^2 + \left(\frac{c_1^3\,c_2}{1728} + \frac{13\,c_1^2\,c_2^2}{4608} + \frac{c_1\,c_2^3}{1728}\right)\hbar^4 + O[\hbar]^5 \\ t[1] \qquad\qquad -\frac{c_2\,\hbar}{8} + \left(-\frac{1}{48}\,c_1\,c_2 - \frac{c_2^2}{16}\right)\hbar^2 + \left(-\frac{7\,c_1^2\,c_2}{1728} - \frac{5}{384}\,c_1 \\ t[2] \qquad\qquad \frac{1}{2} + \left(\frac{c_1}{8} + \frac{c_2}{6}\right)\hbar + \left(\frac{c_1^2}{48} + \frac{c_1\,c_2}{16} + \frac{c_2^2}{24}\right)\hbar^2 + \left(-\frac{37\,c_1^2\,c_2}{5760} + \frac{11}{19}\right. \end{pmatrix}$$

```
C1 = C0 /. sol1
```

$$\begin{pmatrix} 1 - \frac{1}{32}\,c_1^2\,\hbar^2 + \frac{35\,c_1^4\,\hbar^4}{55\,296} + O[\hbar]^5 \\ t[1] \end{pmatrix}$$

```
HardR4[V1]
```

True

```
TwistEq[V1]
```

True

```
V1 ** (V1 // dA[1] // dA[2])
```

(1)

```
CapEquation[V1, C1]
```

True

Φ1 = Φ[V1]

$$\begin{pmatrix} 1 & h[1] \\ t[1] & \left(\frac{5\,c_1\,c_2\,c_3}{1728}+\frac{23\,c_2^2\,c_3}{3456}-\frac{1}{576}\,c_2\,c_3^2\right)\hbar^3+O[\hbar]^5 \\ t[2] & \frac{c_3\,\hbar}{24}+\left(-\frac{311\,c_1^2\,c_3}{17\,280}-\frac{29}{864}\,c_1\,c_2\,c_3-\frac{113\,c_2^2\,c_3}{17\,280}-\frac{23\,c_1\,c_3^2}{1152}-\frac{23\,c_2\,c_3^2}{2880}-\frac{11\,c_3^3}{2880}\right)\hbar^3+O[\hbar]^5 \\ t[3] & \frac{c_2\,\hbar}{24}+\left(-\frac{67\,c_1^2\,c_2}{5760}-\frac{1}{64}\,c_1\,c_2^2-\frac{7\,c_2^3}{5760}+\frac{17\,c_1\,c_2\,c_3}{2880}+\frac{19\,c_2^2\,c_3}{1440}+\frac{59\,c_2\,c_3^2}{5760}\right)\hbar^3+O[\hbar]^5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & h[1] \\ t[1] & \left(\frac{5\,c_1\,c_2\,c_3}{1728}+\frac{23\,c_2^2\,c_3}{3456}-\frac{1}{576}\,c_2\,c_3^2\right)\hbar^3+O[\hbar]^5 & \frac{c_3\,\hbar}{8}+\left(-\frac{67\,c_1^2\,c_3}{3456}-\frac{341\,c}{8}\right. \\ t[2] & \frac{c_3\,\hbar}{24}+\left(-\frac{311\,c_1^2\,c_3}{17\,280}-\frac{29}{864}\,c_1\,c_2\,c_3-\frac{113\,c_2^2\,c_3}{17\,280}-\frac{23\,c_1\,c_3^2}{1152}-\frac{23\,c_2\,c_3^2}{2880}-\frac{11\,c_3^3}{2880}\right)\hbar^3+O[\hbar]^5 & \\ t[3] & \frac{c_2\,\hbar}{24}+\left(-\frac{67\,c_1^2\,c_2}{5760}-\frac{1}{64}\,c_1\,c_2^2-\frac{7\,c_2^3}{5760}+\frac{17\,c_1\,c_2\,c_3}{2880}+\frac{19\,c_2^2\,c_3}{1440}+\frac{59\,c_2\,c_3^2}{5760}\right)\hbar^3+O[\hbar]^5 & -\frac{c_1\,\hbar}{8}+\left(\frac{c_1^3}{192}+\frac{5}{288}\,c_1^2\right. \end{pmatrix}$$

Pentagon[Φ1]

True

Hexagon[+1, Φ1]

True

Hexagon[-1, Φ1]

True

Φ1 ** (Φ1 // dP[3, 2, 1])

(1)

Φ1 ** (Φ1 // dS[1] // dS[2] // dS[3])

(1)

R[1, 2, 1 / 2]

$$\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\,\hbar}{8}+\frac{1}{48}\,c_1^2\,\hbar^2+\frac{1}{384}\,c_1^3\,\hbar^3+\frac{c_1^4\,\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}$$

{R[1, 2, 1 / 2], R[1, 2, 1 / 2] // dS[1] // dS[2]}

$$\left\{\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\,\hbar}{8}+\frac{1}{48}\,c_1^2\,\hbar^2+\frac{1}{384}\,c_1^3\,\hbar^3+\frac{c_1^4\,\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 1 & h[2] \\ t[1] & \frac{1}{2}+\frac{c_1\,\hbar}{8}+\frac{1}{48}\,c_1^2\,\hbar^2+\frac{1}{384}\,c_1^3\,\hbar^3+\frac{c_1^4\,\hbar^4}{3840}+O[\hbar]^5 \end{pmatrix}\right\}$$

(R[1, 3, 1 / 2] ** R[2, 3, 1 / 2] ** V1) == (V1 ** (R[1, 3, 1 / 2] // dΔ[1, 1, 2]))

$$\frac{1}{2}+\hbar\left(\frac{c_1}{8}+\frac{c_2}{4}\right)+\hbar^2\left(\frac{c_1^2}{48}+\frac{5\,c_1\,c_2}{48}+\frac{c_2^2}{12}\right)+\hbar^3\left(\frac{c_1^3}{384}+\frac{5}{192}\,c_1^2\,c_2+\frac{3}{64}\,c_1\,c_2^2+\frac{c_2^3}{48}\right)+$$
$$\hbar^4\left(\frac{c_1^4}{3840}+\frac{19\,c_1^3\,c_2}{3840}+\frac{7}{480}\,c_1^2\,c_2^2+\frac{9}{640}\,c_1\,c_2^3+\frac{c_2^4}{240}\right)==\frac{1}{2}+\hbar\left(\frac{c_1}{8}+\frac{c_2}{8}\right)+\hbar^2\left(\frac{c_1^2}{48}+\frac{c_1\,c_2}{24}+\frac{c_2^2}{48}\right)+$$
$$\hbar^3\left(\frac{c_1^3}{384}+\frac{1}{128}\,c_1^2\,c_2+\frac{1}{128}\,c_1\,c_2^2+\frac{c_2^3}{384}\right)+\hbar^4\left(\frac{c_1^4}{3840}+\frac{1}{960}\,c_1^3\,c_2+\frac{1}{640}\,c_1^2\,c_2^2+\frac{1}{960}\,c_1\,c_2^3+\frac{c_2^4}{3840}\right)\&\&$$
$$\frac{1}{2}+\frac{\hbar\,c_2}{8}+\hbar^2\left(-\frac{c_1^2}{24}-\frac{c_1\,c_2}{48}+\frac{c_2^2}{48}\right)+\hbar^3\left(-\frac{c_1^3}{64}-\frac{1}{32}\,c_1^2\,c_2-\frac{1}{96}\,c_1\,c_2^2+\frac{c_2^3}{384}\right)+$$
$$\hbar^4\left(-\frac{7\,c_1^4}{1920}-\frac{23\,c_1^3\,c_2}{1920}-\frac{11}{960}\,c_1^2\,c_2^2-\frac{11\,c_1\,c_2^3}{3840}+\frac{c_2^4}{3840}\right)==\frac{1}{2}+\hbar\left(\frac{c_1}{8}+\frac{c_2}{8}\right)+\hbar^2\left(\frac{c_1^2}{48}+\frac{c_1\,c_2}{24}+\frac{c_2^2}{48}\right)+$$
$$\hbar^3\left(\frac{c_1^3}{384}+\frac{1}{128}\,c_1^2\,c_2+\frac{1}{128}\,c_1\,c_2^2+\frac{c_2^3}{384}\right)+\hbar^4\left(\frac{c_1^4}{3840}+\frac{1}{960}\,c_1^3\,c_2+\frac{1}{640}\,c_1^2\,c_2^2+\frac{1}{960}\,c_1\,c_2^3+\frac{c_2^4}{3840}\right)$$

```
(V1 // dP[2, 1]) ** Θ[1, 2] == R[1, 2] ** V1
```

True

```
(V1 // dP[2, 1]) ** Θ[1, 2, -1] == R[2, 1, -1] ** V1
```

True

```
(R[1, 3] ** R[2, 3] ** V1) == (V1 ** (R[1, 3] // dΔ[1, 1, 2]))
```

True

```
(R[2, 3, -1] ** R[1, 3, -1] ** V1) == (V1 ** (R[1, 3, -1] // dΔ[1, 1, 2]))
```

True

```
V1 ** (V1 // dS[1] // dS[2])
```

$$\begin{pmatrix} 1 & h[1] \\ t[2] & 1+\frac{c_2\hbar}{2}+\frac{1}{6}c_2^2\hbar^2+\frac{1}{24}c_2^3\hbar^3+\frac{1}{120}c_2^4\hbar^4+O[\hbar]^5 \end{pmatrix}$$

```
R[2, 1]
```

$$\begin{pmatrix} 1 & h[1] \\ t[2] & 1+\frac{c_2\hbar}{2}+\frac{1}{6}c_2^2\hbar^2+\frac{1}{24}c_2^3\hbar^3+\frac{1}{120}c_2^4\hbar^4+O[\hbar]^5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & h[1] \\ t[2] & 1+\frac{c_2\hbar}{2}+\frac{1}{6}c_2^2\hbar^2+\frac{1}{24}c_2^3\hbar^3+O[\hbar]^4 \end{pmatrix}$$

```
False && Put[{V1, sol}, "VToDegree4-120420.m"]
```

False