
Utilities

```

h̄ = 1;
βSimplify[expr_] := expr // Together // ExpandDenominator // ExpandNumerator;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, μ_]] := B[
  βSimplify[ω],
  Collect[μ, _h, Collect[#, _t, βSimplify] &]
];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] → s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | c_s_ → s, Infinity]];
dL[β_] := Union[hL[β], tL[β]];
βForm[B[ω_, μ_]] := Module[
  {tails, heads, mat},
  tails = tL[B[ω, μ]]; heads = hL[B[ω, μ]];
  mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
  PrependTo[mat, t /@ tails];
  mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
  MatrixForm[mat]
];
βForm[else_] := else /. β_B → βForm[β];
Format[β_B, StandardForm] := βForm[β];
B /: B[ω1_, μ1_] == B[ω2_, μ2_] := Module[
  {heads, tails},
  tails = tL[{B[ω1, μ1], B[ω2, μ2]}];
  heads = hL[{B[ω1, μ1], B[ω2, μ2]}];
  (ω1 == ω2) && (
    And @@ Flatten[Outer[
      (Coefficient[μ1, t[#1] h[#2]] == Coefficient[μ2, t[#1] h[#2]]) &,
      tails, heads
    ]]
  )
]

```

The Meta-Cross-Product

The “Tails” meta-group

```

tm[x_, y_, z_][β_] := βCollect[β /. {t[x] → t[z], t[y] → t[z], c_x → c_z, c_y → c_z}];
tΔ[z_, x_, y_][β_] := βCollect[β /. {t[z] → t[x] + t[y], c_z → c_x + c_y}];
tη[x_][β_] := βCollect[(β /. t[x] → 0) /. c_x → 0];
tS[x_][β_] := βCollect[β /. {t[x] → -t[x], c_x → -c_x}];
tA[_][β_] := βCollect[β];
tP[rules___Rule][β_] := βCollect[
  β /. {t[x_] → t[x /. {rules}], c_x → c_x /. {rules}}
];

```

The “Heads” meta-group

```

hm[x_, y_, z_][B[ω_, μ_]] := Module[
  {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
  B[ω, M + h[z] (γx + γy + (γx /. t[i_] → ħ c_i) γy)] // βCollect
];
hΔ[z_, x_, y_][β_] := βCollect[β /. h[z] → h[x] + h[y]];
hη[x_][β_] := βCollect[β /. h[x] → 0];
hS[x_][B[ω_, μ_]] := Module[{γ},
  γ = 1 + D[μ, h[x]] /. t[s_] → ħ c_s;
  βCollect[B[ω, μ /. h[x] → -h[x] / γ]]
];
hA[x_][β_] := hS[x][β];
hP[rules___Rule][β_] := βCollect[β /. h[x_] → h[x /. {rules}]];

```

The TH → HT and HT → TH Swaps

```

thswap[x_, y_][B[ω_, μ_]] := Module[
  {α, β, γ, δ, ε},
  α = Coefficient[μ, h[y] t[x]];
  β = D[μ, t[x]] /. h[y] → 0;
  γ = D[μ, h[y]] /. t[x] → 0;
  δ = μ /. h[y] | t[x] → 0;
  ε = 1 + ħ c_x α;
  B[ω * ε, Plus[
    α (1 + (γ /. t[i_] → ħ c_i) / ε) h[y] t[x],
    β (1 + (γ /. t[i_] → ħ c_i) / ε) t[x],
    γ / ε h[y],
    δ - ħ c_x / ε γ * β
  ]] // βCollect
];
htswap[x_, y_][β_] := β // hS[x] // thswap[y, x] // hS[x];

```

The “double” meta-group

```

dm[x_, y_, z_][β_] := β // thswap[x, y] // hm[x, y, z] // tm[x, y, z];
dΔ[z_, x_, y_][β_] := β // tΔ[z, x, y] // hΔ[z, x, y];
ds[s_][β_] := β // htswap[s, s] // hS[s] // tS[s];
dA[s_][β_] := β // htswap[s, s] // hA[s] // tA[s];
dη[s_][β_] := β // hη[s] // tη[s];
dcap[s_][β_] := β // htswap[s, s] // hη[s];
dP[rules___][β_] := β // hP[rules] // tP[rules];
dP[pl_List][β_] := Module[
  {σ, len, β1, k},
  len = Length[pl];
  β1 = β // (dP @@ Table[i → σ[i], {i, len}]);
  Do[
    k = pl[[i, 1]];
    β1 = β1 // dP[σ[i] → k];
    Do[
      β1 = β1 // dΔ[k, k, pl[[i, j]]],
      {j, 2, Length[pl[[i]]]}
    ],
    {i, len}
  ];
  β1
];
dP[pl___Integer] := dP[IntegerDigits /@ {pl}];

```

The “external” product

```

B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];

```

“Braid-Like” operations

```

Unprotect[NonCommutativeMultiply];
β_ ** ν_ := Module[
  {ρ, σ, labels},
  ρ = β * (ν /. {h[s_] → h[σ[s]], t[s_] → t[σ[s]], c_s_ → c_σ[s]});
  labels = Union[Cases[{β, ν}, h[s_] | t[s_] | c_s_ → s, Infinity]];
  Do[
    ρ = ρ // dm[s, σ[s], s],
    {s, labels}
  ];
  ρ
];
B /: Inverse[B[ω_, μ_]] := Module[
  {ρ = B[1, μ]},
  Do[ρ = ρ // dA[s], {s, dL[ρ]}];
  ReplacePart[ρ, 1 → 1/ω] // βCollect
];

```

The R-Matrix

```

R[x_, y_, p_] :=  $\beta$ Collect[B[1, (E^(p  $\hbar$  c_x) - 1) / ( $\hbar$  c_x) * t[x] h[y]]];
R[x_, y_] := R[x, y, 1];
Ri[x_, y_] := R[x, y, -1];
 $\Theta$ [x_, y_, p_] := (R[x, x, p / 2] // d $\Delta$ [x, x, y]) ** R[x, x, -p / 2] ** R[y, y, -p / 2];
 $\Theta$ [x_, y_] :=  $\Theta$ [x, y, 1];
 $\Theta$ i[x_, y_] :=  $\Theta$ [x, y, -1];

```

Testing the meta-cross-product axioms

The “T” meta-group

```

{
   $\beta$  = B[ $\omega$ [c1, c2, c3, c4], Sum[ $\alpha$ i[c1, c2, c3, c4] t[i] h[1], {i, 4}]],
   $\beta$  // tm[1, 2, 1],
  t1 =  $\beta$  // tm[1, 2, 1] // tm[1, 3, 1],
  t2 =  $\beta$  // tm[2, 3, 28] // tm[1, 28, 1],
  t1 == t2
} //  $\beta$ Form // ColumnForm


$$\begin{pmatrix} \omega[c_1, c_2, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_2, c_3, c_4] \\ t[2] & \alpha_2[c_1, c_2, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_2, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_2, c_3, c_4] \end{pmatrix}$$


$$\begin{pmatrix} \omega[c_1, c_1, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_3, c_4] + \alpha_2[c_1, c_1, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_1, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_3, c_4] \end{pmatrix}$$


$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}$$


$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}$$

True

```

The “H” meta-group

```

{
   $\beta = \mathbf{B}[\omega, \text{Sum}[\alpha_{10\ i+j} t[i] h[j], \{i, 2\}, \{j, 4\}]],$ 
   $\beta // \text{hm}[1, 2, 1],$ 
   $t1 = \beta // \text{hm}[1, 2, 1] // \text{hm}[1, 3, 1],$ 
   $t2 = \beta // \text{hm}[2, 3, 28] // \text{hm}[1, 28, 1],$ 
   $t1 == t2$ 
} //  $\beta\text{Form} // \text{ColumnForm}$ 


$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$$



$$\begin{pmatrix} \omega & & h[1] & & h[3] & h[4] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_2 \alpha_{12} \alpha_{21} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$$



$$\begin{pmatrix} \omega & & & & & h[1] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + \alpha_{13} + c_1 \alpha_{11} \alpha_{13} + c_1 \alpha_{12} \alpha_{13} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{13} + c_2 \alpha_{12} \alpha_{21} + c_2 \alpha_{13} \alpha_{21} + c_1 c_2 \alpha_{12} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} + \alpha_{23} + c_1 \alpha_{11} \alpha_{23} + c_1 \alpha_{12} \alpha_{23} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{23} + c_2 \alpha_{21} \alpha_{23} + c_1 c_2 \alpha_{12} \end{pmatrix}$$



$$\begin{pmatrix} \omega & & & & & h[1] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + \alpha_{13} + c_1 \alpha_{11} \alpha_{13} + c_1 \alpha_{12} \alpha_{13} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{13} + c_2 \alpha_{12} \alpha_{21} + c_2 \alpha_{13} \alpha_{21} + c_1 c_2 \alpha_{12} \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} + \alpha_{23} + c_1 \alpha_{11} \alpha_{23} + c_1 \alpha_{12} \alpha_{23} + c_1^2 \alpha_{11} \alpha_{12} \alpha_{23} + c_2 \alpha_{21} \alpha_{23} + c_1 c_2 \alpha_{12} \end{pmatrix}$$

True

```

```

{
   $\beta = B[\omega, \text{Sum}[\alpha_{10\ i+j}[c_1, c_2] * t[i] h[j], \{i, 2\}, \{j, 2\}]],$ 
   $\beta // t\Delta[2, 2, 3],$ 
   $\beta // h\Delta[2, 2, 3],$ 
   $\beta // h\Delta[2, 2, 3] // hS[3],$ 
   $\beta // h\Delta[2, 2, 3] // hS[3] // hm[2, 3, 2],$ 
   $\beta // h\Delta[2, 2, 3] // hS[3] // hm[3, 2, 2],$ 
   $\beta // hS[1],$ 
   $\beta // hS[1] // hS[1]$ 
} //  $\beta$ Form // ColumnForm


$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2 + c_3] & \alpha_{12}[c_1, c_2 + c_3] \\ t[2] & \alpha_{21}[c_1, c_2 + c_3] & \alpha_{22}[c_1, c_2 + c_3] \\ t[3] & \alpha_{21}[c_1, c_2 + c_3] & \alpha_{22}[c_1, c_2 + c_3] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] & -\frac{\alpha_{12}[c_1, c_2]}{1 + c_1 \alpha_{12}[c_1, c_2] + c_2 \alpha_{22}[c_1, c_2]} \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] & -\frac{\alpha_{22}[c_1, c_2]}{1 + c_1 \alpha_{12}[c_1, c_2] + c_2 \alpha_{22}[c_1, c_2]} \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] \\ t[1] & \alpha_{11}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & -\frac{\alpha_{11}[c_1, c_2]}{1 + c_1 \alpha_{11}[c_1, c_2] + c_2 \alpha_{21}[c_1, c_2]} & \alpha_{12}[c_1, c_2] \\ t[2] & -\frac{\alpha_{21}[c_1, c_2]}{1 + c_1 \alpha_{11}[c_1, c_2] + c_2 \alpha_{21}[c_1, c_2]} & \alpha_{22}[c_1, c_2] \end{pmatrix}$$


$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$$


```

```

{
   $\beta = \mathbf{B}[\omega, \text{Sum}[\alpha_{10\ i+j} * t[i] h[j], \{i, 2\}, \{j, 3\}]],$ 
  t1 =  $\beta$  // hm[1, 2, 1] // hS[1],
  t2 =  $\beta$  // hS[1] // hS[2] // hm[2, 1, 1],
  t1 == t2 // Simplify
} //  $\beta$ Form // ColumnForm


$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$



$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \frac{-\alpha_{11}-\alpha_{12}-c_1 \alpha_{11} \alpha_{12}-c_2 \alpha_{12} \alpha_{21}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & \alpha_{13} \\ t[2] & \frac{-\alpha_{21}-\alpha_{22}-c_1 \alpha_{11} \alpha_{22}-c_2 \alpha_{21} \alpha_{22}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & \alpha_{23} \end{pmatrix}$$



$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \frac{-\alpha_{11}-\alpha_{12}-c_1 \alpha_{11} \alpha_{12}-c_2 \alpha_{12} \alpha_{21}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & \alpha_{13} \\ t[2] & \frac{-\alpha_{21}-\alpha_{22}-c_1 \alpha_{11} \alpha_{22}-c_2 \alpha_{21} \alpha_{22}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & \alpha_{23} \end{pmatrix}$$


```

True

Testing "thswap"

```

Clear[ $\beta$ ];
 $\beta1 = \mathbf{B}[\omega, h[1] t[1] \alpha + h[2] t[1] \beta + h[1] t[2] \gamma + h[2] t[2] \delta],$ 
 $\beta1$  // thswap[1, 1]
} //  $\beta$ Form

```

$$\left\{ \begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \end{pmatrix}, \begin{pmatrix} \omega + \alpha \omega c_1 & h[1] & h[2] \\ t[1] & \frac{\alpha + \alpha^2 c_1 + \alpha \gamma c_2}{1 + \alpha c_1} & \frac{\beta + \alpha \beta c_1 + \beta \gamma c_2}{1 + \alpha c_1} \\ t[2] & \frac{\gamma}{1 + \alpha c_1} & \frac{\delta - \beta \gamma c_1 + \alpha \delta c_1}{1 + \alpha c_1} \end{pmatrix} \right\}$$

```

{
   $\beta = B[\omega, \text{Sum}[\alpha_{10\ i+j} t[i] h[j], \{i, 2\}, \{j, 3\}]],$ 
   $\beta // \text{hm}[1, 2, 1],$ 
   $t1 = \beta // \text{hm}[1, 2, 1] // \text{thswap}[1, 1],$ 
   $t2 = \beta // \text{thswap}[1, 1] // \text{thswap}[1, 2] // \text{hm}[1, 2, 1],$ 
   $t1 = t2 // \text{Simplify}$ 
} //  $\beta\text{Form} // \text{ColumnForm}$ 


$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$


$$\begin{pmatrix} \omega & & h[1] & & h[3] \\ t[1] & \alpha_{11} + \alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_2 \alpha_{12} \alpha_{21} & \alpha_{13} & & \\ t[2] & \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_2 \alpha_{21} \alpha_{22} & \alpha_{23} & & \end{pmatrix}$$


$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{12} + \omega c_1^2 \alpha_{11} \alpha_{12} + \omega c_1 c_2 \alpha_{12} \alpha_{21} & & & & \\ & t[1] & & & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{12} + 3 c_1 c_2 \alpha_{11} \alpha_{12} + 2 c_1^2 \alpha_{11}^2 \alpha_{12} + c_1 \alpha_{12}^2 + 2 c_1^2 \alpha_{11} \alpha_{12}^2 + c_1^3 \alpha_{11}^2 \alpha_{12}^2 + c_2}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \\ & t[2] & & & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{12} + 3 c_1 c_2 \alpha_{11} \alpha_{12} + 2 c_1^2 \alpha_{11}^2 \alpha_{12} + c_1 \alpha_{12}^2 + 2 c_1^2 \alpha_{11} \alpha_{12}^2 + c_1^3 \alpha_{11}^2 \alpha_{12}^2 + c_2}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \end{pmatrix}$$


$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{12} + \omega c_1^2 \alpha_{11} \alpha_{12} + \omega c_1 c_2 \alpha_{12} \alpha_{21} & & & & \\ & t[1] & & & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{12} + 3 c_1 c_2 \alpha_{11} \alpha_{12} + 2 c_1^2 \alpha_{11}^2 \alpha_{12} + c_1 \alpha_{12}^2 + 2 c_1^2 \alpha_{11} \alpha_{12}^2 + c_1^3 \alpha_{11}^2 \alpha_{12}^2 + c_2}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \\ & t[2] & & & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{12} + 3 c_1 c_2 \alpha_{11} \alpha_{12} + 2 c_1^2 \alpha_{11}^2 \alpha_{12} + c_1 \alpha_{12}^2 + 2 c_1^2 \alpha_{11} \alpha_{12}^2 + c_1^3 \alpha_{11}^2 \alpha_{12}^2 + c_2}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \end{pmatrix}$$

True

```

```

{
   $\beta = B[\omega, \text{Sum}[\alpha_{10\ i+j} t[i] h[j], \{i, 3\}, \{j, 2\}]],$ 
   $t1 = \beta // \text{tm}[1, 2, 1] // \text{thswap}[1, 1],$ 
   $t2 = \beta // \text{thswap}[2, 1] // \text{thswap}[1, 1] // \text{tm}[1, 2, 1],$ 
   $t1 = t2 // \text{Simplify}$ 
} //  $\beta\text{Form} // \text{ColumnForm}$ 


$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \end{pmatrix}$$


$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{21} & & h[1] & & h[2] \\ & t[1] & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{21} + 2 c_1 \alpha_{11} \alpha_{21} + c_1 \alpha_{21}^2 + c_3 \alpha_{11} \alpha_{31} + c_3 \alpha_{21} \alpha_{31}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} & & \frac{\alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_1 \alpha_{12} \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_1 \alpha_{21} \alpha_{22} + c_3 \alpha_{11} \alpha_{32} + c_3 \alpha_{21} \alpha_{32}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \\ & t[3] & \frac{\alpha_{31}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} & & \frac{-c_1 \alpha_{12} \alpha_{31} - c_1 \alpha_{22} \alpha_{31} + \alpha_{32} + c_1 \alpha_{11} \alpha_{32} + c_1 \alpha_{21} \alpha_{32}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \end{pmatrix}$$


$$\begin{pmatrix} \omega + \omega c_1 \alpha_{11} + \omega c_1 \alpha_{21} & & h[1] & & h[2] \\ & t[1] & \frac{\alpha_{11} + c_1 \alpha_{11}^2 + \alpha_{21} + 2 c_1 \alpha_{11} \alpha_{21} + c_1 \alpha_{21}^2 + c_3 \alpha_{11} \alpha_{31} + c_3 \alpha_{21} \alpha_{31}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} & & \frac{\alpha_{12} + c_1 \alpha_{11} \alpha_{12} + c_1 \alpha_{12} \alpha_{21} + \alpha_{22} + c_1 \alpha_{11} \alpha_{22} + c_1 \alpha_{21} \alpha_{22} + c_3 \alpha_{11} \alpha_{32} + c_3 \alpha_{21} \alpha_{32}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \\ & t[3] & \frac{\alpha_{31}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} & & \frac{-c_1 \alpha_{12} \alpha_{31} - c_1 \alpha_{22} \alpha_{31} + \alpha_{32} + c_1 \alpha_{11} \alpha_{32} + c_1 \alpha_{21} \alpha_{32}}{1 + c_1 \alpha_{11} + c_1 \alpha_{21}} \end{pmatrix}$$

True

```


Testing "htswap"

```
Clear[β];
{β1 = B[ω, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // htswap[1, 1]
} // βForm
```

$$\left\{ \begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha & \beta \\ t[2] & \gamma & \delta \end{pmatrix}, \begin{pmatrix} \frac{\omega+\gamma\omega c_2}{1+\alpha c_1+\gamma c_2} & h[1] & h[2] \\ t[1] & \frac{\alpha}{1+\gamma c_2} & \frac{\beta}{1+\gamma c_2} \\ t[2] & \frac{\gamma+\alpha\gamma c_1+\gamma^2 c_2}{1+\gamma c_2} & \frac{\delta+\beta\gamma c_1+\gamma\delta c_2}{1+\gamma c_2} \end{pmatrix} \right\}$$

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 3}]],
  t1 = β // hm[1, 2, 1] // htswap[1, 1],
  t2 = β // htswap[2, 1] // htswap[1, 1] // hm[1, 2, 1],
  t1 == t2 // Simplify
} // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega+\omega c_2 \alpha_{21}+\omega c_2 \alpha_{22}+\omega c_1 c_2 \alpha_{11} \alpha_{22}+\omega c_2^2 \alpha_{21} \alpha_{22}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & t[1] & & \\ & t[2] & & \alpha_{21}+c_1 \alpha_{11} \alpha_{21}+c_1 \alpha_{12} \alpha_{21}+c_1^2 \alpha_{11} \alpha_{12} \alpha_{21}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}^2+\alpha_{22} \\ \frac{\omega+\omega c_2 \alpha_{21}+\omega c_2 \alpha_{22}+\omega c_1 c_2 \alpha_{11} \alpha_{22}+\omega c_2^2 \alpha_{21} \alpha_{22}}{1+c_1 \alpha_{11}+c_1 \alpha_{12}+c_1^2 \alpha_{11} \alpha_{12}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}+c_2 \alpha_{22}+c_1 c_2 \alpha_{11} \alpha_{22}+c_2^2 \alpha_{21} \alpha_{22}} & t[1] & & \\ & t[2] & & \alpha_{21}+c_1 \alpha_{11} \alpha_{21}+c_1 \alpha_{12} \alpha_{21}+c_1^2 \alpha_{11} \alpha_{12} \alpha_{21}+c_2 \alpha_{21}+c_1 c_2 \alpha_{12} \alpha_{21}^2+\alpha_{22} \end{pmatrix}$$

True

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // htswap[1, 1],
  t2 = β // htswap[1, 1] // htswap[1, 2] // tm[1, 2, 1],
  t1 == t2 // Simplify
} // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega+\omega c_3 \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}+c_3 \alpha_{31}} & h[1] & h[2] \\ t[1] & \frac{\alpha_{11}+\alpha_{21}}{1+c_3 \alpha_{31}} & \frac{\alpha_{12}+\alpha_{22}}{1+c_3 \alpha_{31}} \\ t[3] & \frac{\alpha_{31}+c_1 \alpha_{11} \alpha_{31}+c_1 \alpha_{21} \alpha_{31}+c_3 \alpha_{31}^2}{1+c_3 \alpha_{31}} & \frac{c_1 \alpha_{12} \alpha_{31}+c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_3 \alpha_{31} \alpha_{32}}{1+c_3 \alpha_{31}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\omega+\omega c_3 \alpha_{31}}{1+c_1 \alpha_{11}+c_1 \alpha_{21}+c_3 \alpha_{31}} & h[1] & h[2] \\ t[1] & \frac{\alpha_{11}+\alpha_{21}}{1+c_3 \alpha_{31}} & \frac{\alpha_{12}+\alpha_{22}}{1+c_3 \alpha_{31}} \\ t[3] & \frac{\alpha_{31}+c_1 \alpha_{11} \alpha_{31}+c_1 \alpha_{21} \alpha_{31}+c_3 \alpha_{31}^2}{1+c_3 \alpha_{31}} & \frac{c_1 \alpha_{12} \alpha_{31}+c_1 \alpha_{22} \alpha_{31}+\alpha_{32}+c_3 \alpha_{31} \alpha_{32}}{1+c_3 \alpha_{31}} \end{pmatrix}$$

True

The “double” meta-group

```
{β = B[ω, Sum[α10 i+j t[i] h[j], {i, 4}, {j, 4}]],
  t1 = β // dm[1, 2, 1] // dm[1, 3, 1],
  t2 = β // dm[2, 3, 2] // dm[1, 2, 1],
  t1 = t2 // Simplify
} // βForm // ColumnForm
```

A very large output was generated. Here is a sample of it:

$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ t[4] & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$	
$\begin{pmatrix} \omega + \omega c_1 \alpha_{12} + \omega c_1 \alpha_{13} + \omega c_1^2 \alpha_{12} \alpha_{13} + \omega c_1 \alpha_{23} + \omega c_1^2 \alpha_{12} \alpha_{23} + \omega c_1^2 \alpha_{13} \alpha_{32} + \omega c_1 c_4 \alpha_{13} \alpha_{42} \\ t[1] \\ t[4] \end{pmatrix}$	$\begin{matrix} h[1] \\ \frac{\alpha_{11} + \ll 646 \gg + c_1^4 \alpha_{13}}{1 + \ll 6 \gg + c_1 c_4} \\ \ll 1 \gg \\ \ll 1 \gg \end{matrix}$
$\begin{pmatrix} \omega + \omega c_1 \alpha_{12} + \omega c_1 \alpha_{13} + \omega c_1^2 \alpha_{12} \alpha_{13} + \omega c_1 \alpha_{23} + \omega c_1^2 \alpha_{12} \alpha_{23} + \omega c_1^2 \alpha_{13} \alpha_{32} + \omega c_1 c_4 \alpha_{13} \alpha_{42} \\ t[1] \\ t[4] \end{pmatrix}$	$\begin{matrix} h[1] \\ \frac{\alpha_{11} + \ll 646 \gg + c_1^4 \alpha_{13}}{1 + c_1 \alpha_{12} + \ll 5 \gg} \\ \ll 1 \gg \end{matrix}$
True	

Show Less	Show More	Show Full Output	Set Size Limit...
-----------	-----------	------------------	-------------------

The “braid-like” operations

```
{β = B[ω, Sum[α10 i+j [c1, c2] t[i] h[j], {i, 2}, {j, 2}]],
  Inverse[β],
  β ** Inverse[β]
} // βForm // ColumnForm
```

$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11}[c_1, c_2] & \alpha_{12}[c_1, c_2] \\ t[2] & \alpha_{21}[c_1, c_2] & \alpha_{22}[c_1, c_2] \end{pmatrix}$	
$\begin{pmatrix} \frac{1}{\omega} \\ t[1] \\ t[2] \end{pmatrix}$	$\begin{matrix} h[1] \\ \frac{-\alpha_{11}[c_1, c_2] - c_1 \alpha_{11}[c_1, c_2] \alpha_{12}[c_1, c_2] - c_2 \alpha_{12}[c_1, c_2] \alpha_{21}[c_1, c_2]}{1 + c_1 \alpha_{11}[c_1, c_2] + c_1 \alpha_{12}[c_1, c_2] + c_1^2 \alpha_{11}[c_1, c_2] \alpha_{12}[c_1, c_2] + 2 c_2 \alpha_{21}[c_1, c_2] + c_1 c_2 \alpha_{11}[c_1, c_2] \alpha_{21}[c_1, c_2] + c_1 c_2 \alpha_{12}[c_1, c_2] \alpha_{21}[c_1, c_2]} \\ - \frac{\alpha_{21}[c_1, c_2]}{1 + c_1 \alpha_{12}[c_1, c_2] + c_2 \alpha_{21}[c_1, c_2]} \end{matrix}$
$\begin{pmatrix} 1 & h[1] & h[2] \\ t[1] & 0 & 0 \\ t[2] & 0 & 0 \end{pmatrix}$	

Some Knot-Theoretic Definitions

```

HardR4[V_] := (R[1, 3] ** R[2, 3] ** V) == (V ** (R[1, 3] // dΔ[1, 1, 2]));
TwistEq[V_] := (V // dP[2, 1]) ** Θ[1, 2] == R[1, 2] ** V;
CapEquation[V_, Cap_] := (V ** (Cap // dP[12]) // dcap[1] // dcap[2]) ==
  (Cap (Cap // dP[2]) // dcap[1] // dcap[2]);
Φ[V_] := (Inverse[V] // dP[12, 3]) ** Inverse[V] ** (V // dP[2, 3]) ** (V // dP[1, 23]);
Pentagon[Φ_] := Φ ** (Φ // dP[1, 23, 4]) ** (Φ // dP[2, 3, 4]) ==
  (Φ // dP[12, 3, 4]) ** (Φ // dP[1, 2, 34]);
Hexagon[s_, Φ_] := Equal[
  Θ[1, 2, s] // dP[12, 3],
  Φ ** Θ[2, 3, s] ** Inverse[Φ // dP[1, 3, 2]] ** Θ[1, 3, s] ** (Φ // dP[3, 1, 2])
];
Rot120[β_] := β // dS[2] // dΔ[2, 2, 3] // dμ[1, 3, 1] // dP[2, 1];
{β = B[ω[c1, c2], Sum[α10 i+j[c1, c2] t[i] h[j], {i, 2}, {j, 2}]],
  β // Rot120,
  β // Rot120 // Rot120,
  β // Rot120 // Rot120 // Rot120
} // βForm // ColumnForm

(
  ω[c1, c2]      h[1]      h[2]
  t[1]      α11[c1, c2]  α12[c1, c2]
  t[2]      α21[c1, c2]  α22[c1, c2]
)

(
  ω[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]-c1 α22[c2, -c1-c2]-c2 α22[c2, -c1-c2]
  t[1]
  t[2]
)

(
  ω[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
  t[1]
  t[2]
)

(
  ω[c1, c2]      h[1]      h[2]
  t[1]      α11[c1, c2]  α12[c1, c2]
  t[2]      α21[c1, c2]  α22[c1, c2]
)

(
  h[1]
  α22[c2, -c1-c2]
  -1-c2 α12[c2, -c1-c2]+c1 α22[c2, -c1-c2]+c2 α22[c2, -c1-c2]
  -α12[c2, -c1-c2]+α22[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]-c1 α22[c2, -c1-c2]-c2 α22[c2, -c1-c2]
  α11[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]
)

(
  h[1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]+α21[-c1-c2, c1]-α22[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
)

(
  h[1]
  -1-c2 α12[c2, -c1-c2]
  -1-c2 α12[c2, -c1-c2]+c1 α22[c2, -c1-c2]+c2 α22[c2, -c1-c2]
  α11[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]-c1 α22[c2, -c1-c2]-c2 α22[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]
)

(
  h[1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]+α21[-c1-c2, c1]-α22[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
)

(
  h[1]
  -1-c2 α12[c2, -c1-c2]
  -1-c2 α12[c2, -c1-c2]+c1 α22[c2, -c1-c2]+c2 α22[c2, -c1-c2]
  α11[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]-c1 α22[c2, -c1-c2]-c2 α22[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]
)

(
  h[1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]+α21[-c1-c2, c1]-α22[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
)

(
  h[1]
  -1-c2 α12[c2, -c1-c2]
  -1-c2 α12[c2, -c1-c2]+c1 α22[c2, -c1-c2]+c2 α22[c2, -c1-c2]
  α11[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]-c1 α22[c2, -c1-c2]-c2 α22[c2, -c1-c2]
  1+c2 α12[c2, -c1-c2]
)

(
  h[1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]+α21[-c1-c2, c1]-α22[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
  -α11[-c1-c2, c1]+α12[-c1-c2, c1]
  -1+c1 α11[-c1-c2, c1]+c2 α11[-c1-c2, c1]-c1 α21[-c1-c2, c1]
)

```