Pensieve header: Perturbative β-calculations.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-03"];
<< betaCalculus.m
```

```
Clear[ħ];
$PerturbativeDegree = 4;
βSimplify[expr_] := Replace[
   Series[Normal[expr], {ħ, 0, $PerturbativeDegree}],
   sd_SeriesData :> MapAt[Expand, sd, 3]
  ];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   βSimplify[μ]
  ];
```

The Knot-Theoretic Equations

R2, OC, R3 and easy R4

```
{R[1, 2] Ri[3, 4],
  R[1, 2] Ri[3, 4] // dm[1, 3, 1] // dm[2, 4, 2],
  R[1, 2] Ri[3, 4] // dm[1, 3, 1] // dm[4, 2, 2]
 } // ColumnForm
```

$$\begin{pmatrix} 1 & h[2] & h[4] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 & 0 \\ t[3] & 0 & -1+\frac{c_3\hbar}{2}-\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3-\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 \end{pmatrix}$$

$$(1)$$

$$(1)$$

```
{
  R[1, 2] ** Ri[1, 2],
  R[1, 3] ** R[2, 3],
  R[1, 3] ** R[2, 3] == R[2, 3] ** R[1, 3] // Expand ,
  R[3, 1] ** R[3, 2] == R[3, 2] ** R[3, 1],
  R[1, 2] ** R[1, 3] ** R[2, 3],
  R[1, 2] ** R[1, 3] ** R[2, 3] == R[2, 3] ** R[1, 3] ** R[1, 2]
 } // ColumnForm
```

$$(1)$$

$$\begin{pmatrix} 1 & h[3] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 \\ t[2] & 1+\left(c_1+\frac{c_2}{2}\right)\hbar+\left(\frac{c_1^2}{2}+\frac{c_1c_2}{2}+\frac{c_2^2}{6}\right)\hbar^2+\left(\frac{c_1^3}{6}+\frac{1}{4}c_1^2c_2+\frac{1}{6}c_1c_2^2+\frac{c_2^3}{24}\right)\hbar^3+\left(\frac{c_1^4}{24}+\frac{1}{12}c_1^3c_2+\frac{1}{12}c_1^2c_2^2+\frac{1}{24}\right. \end{pmatrix}$$

$$1+\frac{\hbar c_1}{2}+\frac{1}{6}\hbar^2c_1^2+\frac{1}{24}\hbar^3c_1^3+\frac{1}{120}\hbar^4c_1^4 == 1+\frac{\hbar c_1}{2}+\frac{1}{6}\hbar^2c_1^2+\frac{1}{24}\hbar^3c_1^3+\frac{1}{120}\hbar^4c_1^4+\hbar c_2+\frac{1}{2}\hbar^2c_1c_2+\frac{1}{6}\hbar^3c_1^2c$$

True

$$\begin{pmatrix} 1 & h[2] & \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2 \\ t[2] & 0 & 1+\left(c_1+\frac{c_2}{2}\right)\hbar+\left(\frac{c_1^2}{2}+\frac{c_1c_2}{2}+\frac{c_2^2}{6}\right)\hbar^2+\left(\frac{c_1^3}{6}+\frac{1}{4}c_1^2c_2+ \end{pmatrix}$$

True

```
{
 R[3, 1] ** R[3, 2],
 R[3, 1],
 R[3, 1] // dΔ[1, 1, 2],
 R[3, 1] ** R[3, 2] == (R[3, 1] // dΔ[1, 1, 2])
}
```

$$\left\{\begin{pmatrix} 1 & h[1] & h[2] \\ t[3] & 1+\frac{c_3\hbar}{2}+\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3+\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 & 1+\frac{c_3\hbar}{2}+\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3+\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 \end{pmatrix},\right.$$
$$\begin{pmatrix} 1 & h[1] \\ t[3] & 1+\frac{c_3\hbar}{2}+\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3+\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 \end{pmatrix},$$
$$\left.\begin{pmatrix} 1 & h[1] & h[2] \\ t[3] & 1+\frac{c_3\hbar}{2}+\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3+\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 & 1+\frac{c_3\hbar}{2}+\frac{1}{6}c_3^2\hbar^2+\frac{1}{24}c_3^3\hbar^3+\frac{1}{120}c_3^4\hbar^4+O[\hbar]^5 \end{pmatrix},\right.$$
$$\left.\text{True}\right\}$$

```
R[1, 2, p1] ** R[1, 2, p2] == R[1, 2, p1 + p2] // Simplify
```

```
True
```

Hard R4

```
{
 R[1, 3] ** R[2, 3],
 R[1, 3] // dΔ[1, 1, 2],
 R[1, 3] ** R[2, 3] == (R[1, 3] // dΔ[1, 1, 2])
}
```

$$\left\{\begin{pmatrix} 1 & h[3] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 \\ t[2] & 1+\left(c_1+\frac{c_2}{2}\right)\hbar+\left(\frac{c_1^2}{2}+\frac{c_1c_2}{2}+\frac{c_2^2}{6}\right)\hbar^2+\left(\frac{c_1^3}{6}+\frac{1}{4}c_1^2c_2+\frac{1}{6}c_1c_2^2+\frac{c_2^3}{24}\right)\hbar^3+\left(\frac{c_1^4}{24}+\frac{1}{12}c_1^3c_2+\frac{1}{12}c_1^2c_2^2+\frac{1}{2}\right. \end{pmatrix}\right.$$

$$\left.\hbar^3\left(\frac{c_1^3}{24}+\frac{1}{8}c_1^2c_2+\frac{1}{8}c_1c_2^2+\frac{c_2^3}{24}\right)+\hbar^4\left(\frac{c_1^4}{120}+\frac{1}{30}c_1^3c_2+\frac{1}{20}c_1^2c_2^2+\frac{1}{30}c_1c_2^3+\frac{c_2^4}{120}\right)\right\}\ \&\&$$

$$\left\{\begin{pmatrix} 1 & h[3] \\ t[1] & 1+\frac{c_1\hbar}{2}+\frac{1}{6}c_1^2\hbar^2+\frac{1}{24}c_1^3\hbar^3+\frac{1}{120}c_1^4\hbar^4+O[\hbar]^5 \\ t[2] & 1+\left(c_1+\frac{c_2}{2}\right)\hbar+\left(\frac{c_1^2}{2}+\frac{c_1c_2}{2}+\frac{c_2^2}{6}\right)\hbar^2+\left(\frac{c_1^3}{6}+\frac{1}{4}c_1^2c_2+\frac{1}{6}c_1c_2^2+\frac{c_2^3}{24}\right)\hbar^3+ \\ & \left.\hbar^4\left(\frac{c_1^4}{24}+\frac{1}{12}c_1^3c_2+\frac{1}{12}c_1^2c_2^2+\frac{1}{24}c_1c_2^3+\frac{c_2^4}{120}\right)\right\} == \end{pmatrix}$$

$$1+\hbar\left(c_1+\frac{c_2}{2}\right)+\hbar^2\left(\frac{c_1^2}{2}+\frac{c_1c_2}{2}+\frac{c_2^2}{6}\right)+\hbar^3\left(\frac{c_1^3}{6}+\frac{1}{4}c_1^2c_2+\frac{1}{6}c_1c_2^2+\frac{c_2^3}{24}\right)+$$
$$\hbar^4\left(\frac{c_1^4}{24}+\frac{1}{12}c_1^3c_2+\frac{1}{12}c_1^2c_2^2+\frac{1}{24}c_1c_2^3+\frac{c_2^4}{120}\right)==1+\hbar\left(\frac{c_1}{2}+\frac{c_2}{2}\right)+\hbar^2\left(\frac{c_1^2}{6}+\frac{c_1c_2}{3}+\frac{c_2^2}{6}\right)+$$
$$\hbar^3\left(\frac{c_1^3}{24}+\frac{1}{8}c_1^2c_2+\frac{1}{8}c_1c_2^2+\frac{c_2^3}{24}\right)+\hbar^4\left(\frac{c_1^4}{120}+\frac{1}{30}c_1^3c_2+\frac{1}{20}c_1^2c_2^2+\frac{1}{30}c_1c_2^3+\frac{c_2^4}{120}\right)$$

```
{
  V0 = B[V0[ħ c1, ħ c2], Sum[V10 i+j[ħ c1, ħ c2] t[i] h[j], {i, 2}, {j, 2}]],
  R[1, 3] ** R[2, 3] ** V0,
  V0 ** (R[1, 3] // dΔ[1, 1, 2]),
  eqns1 = HardR4[V0]
 } // ColumnForm
```

$$\begin{pmatrix} V_0[\hbar c_1, \hbar c_2] \\ t[1] \quad V_{11}[0,0] + \left(c_2 V_{11}^{(0,1)}[0,0] + c_1 V_{11}^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_{11}^{(0,2)}[0,0] + c_1 c_2 V_{11}^{(1,} \\ t[2] \quad V_{21}[0,0] + \left(c_2 V_{21}^{(0,1)}[0,0] + c_1 V_{21}^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_{21}^{(0,2)}[0,0] + c_1 c_2 V_{21}^{(1,} \end{pmatrix}$$

$$\begin{pmatrix} V_0[0,0] + \left(c_2 V_0^{(0,1)}[0,0] + c_1 V_0^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_0^{(0,2)}[0,0] + c_1 c_2 V_0^{(1,1)}[0,0] + \frac{1}{2} c_1^2 V_0^{(2,} \end{pmatrix}$$

$$\begin{pmatrix} V_0[0,0] + \left(c_2 V_0^{(0,1)}[0,0] + c_1 V_0^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_0^{(0,2)}[0,0] + c_1 c_2 V_0^{(1,1)}[0,0] + \frac{1}{2} c_1^2 V_0^{(2,} \end{pmatrix}$$

$$\hbar == 0 \,||\, (c_1 == 0 \,\&\&\, c_2 == 0) \,||\, 60\,(1 + 2\,V_{12}[0,0] - 2\,V_{21}[0,0]) + 20\,\hbar\,c_2\left(1 + 3\,V_{12}[0,0] + 6\,V_{12}^{(0,1)}[0,\right.$$

```
sol = Solve[eqns1 && V21[c1, c2] == 0, V12[c1, c2]]
```

```
{}
```

```
V1 = V0 /. {V21[c1, c2] → 0, V11[c1, c2] → 0, V22[c1, c2] → 0, V0[c1, c2] → 1} /. sol[[1]]
```

Part::partw : Part 1 of {} does not exist. ≫

ReplaceAll::reps : {{}[[1]]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. ≫

Part::partw : Part 1 of {} does not exist. ≫

ReplaceAll::reps : {{}[[1]]} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. ≫

$$\begin{pmatrix} V_0[\hbar c_1, \hbar c_2] \\ t[1] \quad V_{11}[0,0] + \left(c_2 V_{11}^{(0,1)}[0,0] + c_1 V_{11}^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_{11}^{(0,2)}[0,0] + c_1 c_2 V_{11}^{(1,} \\ t[2] \quad V_{21}[0,0] + \left(c_2 V_{21}^{(0,1)}[0,0] + c_1 V_{21}^{(1,0)}[0,0]\right)\hbar + \left(\frac{1}{2} c_2^2 V_{21}^{(0,2)}[0,0] + c_1 c_2 V_{21}^{(1,} \end{pmatrix}$$

$$\begin{aligned}
&\textbf{sol2} = \Big\{V_{11}[c_1, c_2] \to \\
&\quad -\Big(\big(-1+e^{\frac{c_2}{2}}\big)\big(e^{\frac{c_1}{2}}c_1 - e^{\frac{c_1}{2}+c_2}c_1 - 2e^{\frac{c_1}{2}+\frac{c_2}{2}}c_2 + e^{c_1+\frac{c_2}{2}}c_2 + e^{\frac{c_2}{2}}c_2 - 2e^{\frac{c_1}{2}+c_2}c_2 + 2e^{c_1+c_2}c_2\big)\Big)\Big/ \\
&\qquad \Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big),\ V_{12}[c_1, c_2] \to \\
&\quad \big(e^{\frac{c_1}{2}}c_1 - e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big/\Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big), \\
&\quad V_{21}[c_1, c_2] \to \big(e^{\frac{c_1}{2}}c_1 - 2e^{c_1+\frac{c_2}{2}}c_1 - e^{\frac{c_1}{2}+c_2}c_1 + 2e^{c_1+\frac{3c_2}{2}}c_1 - e^{c_1+\frac{c_2}{2}}c_2 + e^{\frac{c_2}{2}}c_2\big)\Big/ \\
&\qquad \Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big),\ V_{22}[c_1, c_2] \to 0,\ V_{11}[c_2, c_1] \to \\
&\quad \Big(\big(-1+e^{\frac{c_1}{2}}\big)\big(-e^{\frac{c_1}{2}}c_1 + 2e^{\frac{c_1}{2}+\frac{c_2}{2}}c_1 + 2e^{c_1+\frac{c_2}{2}}c_1 - e^{\frac{c_1}{2}+c_2}c_1 - 2e^{c_1+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big)\Big/ \\
&\qquad \Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big),\ V_{12}[c_2, c_1] \to \\
&\quad -\big(e^{\frac{c_1}{2}}c_1 - e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big/\Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big), \\
&\quad V_{21}[c_2, c_1] \to -\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2 + 2e^{\frac{c_1}{2}+c_2}c_2 - 2e^{\frac{3c_1}{2}+c_2}c_2\big)\Big/ \\
&\qquad \Big((c_1+c_2)\big(-e^{\frac{c_1}{2}}c_1 + e^{\frac{c_1}{2}+c_2}c_1 + e^{c_1+\frac{c_2}{2}}c_2 - e^{\frac{c_2}{2}}c_2\big)\Big),\ V_{22}[c_2, c_1] \to 0\Big\}\ \texttt{/.}\ c_{i_} :\!\to \hbar\, c_i;
\end{aligned}$$

V2 = βCollect[V0 /. sol2 /. V₀[__] → 1]

	1	h[1]
t[1]		$-\frac{c_2\hbar}{8} + \left(-\frac{1}{96}c_1c_2 - \frac{5c_2^2}{96}\right)\hbar^2 + \left(-\frac{1}{192}c_1c_2^2 - \frac{5c_2^3}{384}\right)\hbar^3 + \left(\frac{c_1^3c_2}{11520} - \frac{c_1^2c_2^2}{11520} - \frac{19c_1c_2^3}{11520} - \frac{13}{576}\right.$ …
t[2]	$\frac{1}{2} + \left(\frac{5c_1}{48} + \frac{7c_2}{48}\right)\hbar + \left(\frac{c_1^2}{96} + \frac{c_1c_2}{24} + \frac{c_2^2}{32}\right)\hbar^2 + \left(\frac{c_1^3}{5760} + \frac{29c_1^2c_2}{5760} + \frac{61c_1c_2^2}{5760} + \frac{29c_2^3}{5760}\right)\hbar^3 + \left(-\frac{c_1^4}{11520} + \frac{c_1^3c_2}{5760} + \frac{1}{640}c_1^2\right.$ …	

HardR4[V2]

True

Simplify[(V2 // dP[2, 1]) ** Θ[1, 2] == R[1, 2] ** V2]

True

Simplify[(V2 // dP[2, 1]) ** Θ[1, 2] == R[2, 1] ** V2]

$$\begin{aligned}
&\hbar c_2\left(-\hbar^2c_1^3 + \hbar^2c_1^2c_2 + c_1\left(120 + 120\hbar c_2 + 79\hbar^2c_2^2\right) + 2c_2\left(300 + 300\hbar c_2 + 163\hbar^2c_2^2\right)\right) == 0\ \&\& \\
&\ 5760 + 67\hbar^4c_1^4 + 2880\hbar c_2 + 840\hbar^2c_2^2 + 180\hbar^3c_2^3 + 29\hbar^4c_2^4 + 4\hbar^3c_1^3(75 + 58\hbar c_2) + \\
&\quad 12\hbar^2c_1^2\left(90 + 65\hbar c_2 + 24\hbar^2c_2^2\right) + 4\hbar c_1\left(720 + 480\hbar c_2 + 165\hbar^2c_2^2 + 38\hbar^3c_2^3\right) == 0\ \&\& \\
&\ \hbar^4c_1^4 == 5760 + 5760\hbar c_2 + 2\hbar^4c_1^3c_2 + 3240\hbar^2c_2^2 + 1320\hbar^3c_2^3 + 427\hbar^4c_2^4 + \\
&\quad 6\hbar^2c_1^2\left(20 + 20\hbar c_2 + 13\hbar^2c_2^2\right) + 2\hbar^2c_1c_2\left(240 + 240\hbar c_2 + 131\hbar^2c_2^2\right)\ \&\& \\
&\ \hbar c_1\left(2880 + 251\hbar^3c_1^3 + 840\hbar c_2 + 180\hbar^2c_2^2 + 29\hbar^3c_2^3 + \right. \\
&\qquad \left.\hbar^2c_1^2(840 + 289\hbar c_2) + \hbar c_1\left(2040 + 660\hbar c_2 + 151\hbar^2c_2^2\right)\right) == 0
\end{aligned}$$

Φ2 = Φ[V2]

	1	h[1]	
t[1]		$\left(-\frac{1}{768}c_1c_2c_3 + \frac{7c_2^2c_3}{2304} - \frac{c_2c_3^2}{1152}\right)\hbar^3 + O[\hbar]^5$	$\frac{5c_3\hbar}{48} + \left(-\frac{1}{640}c_1^2c_3 - \frac{11c_1c_2c_3}{2880} - \right.$ …
t[2]	$\frac{c_3\hbar}{48} + \left(-\frac{c_1^2c_3}{5760} + \frac{7c_1c_2c_3}{5760} - \frac{c_2^2c_3}{1920} + \frac{7c_1c_3^2}{11520} - \frac{c_2c_3^2}{1920} - \frac{7c_3^3}{11520}\right)\hbar^3 + O[\hbar]^5$		$\left(-\frac{5c_1^2c}{1152}\right.$ …
t[3]	$\frac{c_2\hbar}{24} + \left(\frac{c_1^2c_2}{5760} - \frac{17c_1c_2^2}{5760} - \frac{7c_2^3}{5760} + \frac{11c_1c_2c_3}{3840} + \frac{c_2^2c_3}{1920} - \frac{1}{640}c_2c_3^2\right)\hbar^3 + O[\hbar]^5$	$-\frac{5c_1\hbar}{48} + \left(\frac{23c_1^3}{11520} + \frac{79c_1^2c_2}{11520} + \frac{17c_1}{5760}\right.$ …	

Pentagon[Φ2]

True

```
Hexagon[+1, Φ2]
```

```
True
```

```
Hexagon[-1, Φ2]
```

```
True
```