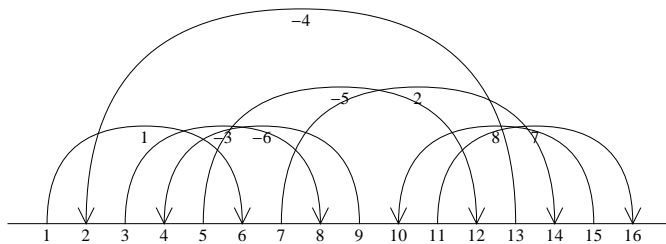


Pensieve Header: Finding the MVA in the  $\beta$ -framework.

```
<< KnotTheory`
GD[K_] := GD @@ (
  PD[K] /. X[i_, j_, k_, l_] => If[PositiveQ[X[i, j, k, l]],
    Ar[l, i, +1], Ar[j, i, -1]
  ]
)
Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.
Draw[expr_] := expr /. gd_GD => Draw[gd];
Draw[gd_GD] := Module[
  {n = Length[gd], h, k = 0},
  Graphics[{
    Line[{{0, 0}, {2 n + 1, 0}}],
    Table[Text[i, {i, -0.3}], {i, 2 n}],
    (List @@ gd) /. {
      Ar[i_, j_, s_] => {
        h = Abs[i - j] / 2;
        BezierCurve[
          {
            {i, 0}, {i, h}, {(i + j) / 2, h}, {j, h}, {j, 0}
          }, SplineDegree -> 2],
        Text[s * (++k), {(i + j) / 2, h - 0.3}],
        Line[{{j - 0.2, 0.4}, {j, 0}, {j + 0.2, 0.4}}]
      }
    }
  ]
];
Draw[GD[Knot[8, 17]]]
```

KnotTheory::loading: Loading precomputed data in PD4Knots`.



```

βSimplify = Factor;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, μ_]] := B[
  βSimplify[ω],
  Collect[μ, _h, Collect[#, _t, βSimplify] &]
];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] => s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | T_s_ => s, Infinity]];
dL[β_] := Union[hL[β], tL[β]];
SetAttributes[βForm, Listable];
βForm[B[ω_, μ_]] := Module[
  {tails, heads, mat},
  tails = tL[B[ω, μ]]; heads = hL[B[ω, μ]];
  mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
  PrependTo[mat, t /@ tails];
  mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
  MatrixForm[mat]
];
R[x_, y_] := B[1, (T_x - 1) t[x] h[y]];
Rinv[x_, y_] := B[1, (1 / T_x - 1) t[x] h[y]];
tm[x_, y_, z_][β_] := β /. {t[x] -> t[z], t[y] -> t[z], T_x -> T_z, T_y -> T_z};
hm[x_, y_, z_][B[ω_, μ_]] := Module[
  {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] -> 0},
  B[ω, M + h[z] (γx + γy + (γx /. t[i_] -> 1) γy)] // βCollect
];
swap[x_, y_][B[ω_, μ_]] := Module[
  {α, β, γ, δ, ε},
  α = Coefficient[μ, h[x] t[y]];
  β = D[μ, t[y]] /. h[x] -> 0;
  γ = D[μ, h[x]] /. t[y] -> 0;
  δ = μ /. h[x] | t[y] -> 0;
  ε = 1 + α;
  B[ω * ε, Plus[
    α (1 + (γ /. t[i_] -> 1) / ε) h[x] t[y],
    β (1 + (γ /. t[i_] -> 1) / ε) t[y],
    γ / ε h[x],
    δ - (1 / ε) γ * β
  ]] // βCollect
];
gm[x_, y_, z_][β_] := β // swap[y, x] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];

```

```

{
  L = Link["L9n1"], skel = Skeleton[L],
  MultivariableAlexander[L][X] // Factor, Draw[GD[L]],
  (β = Times @@ GD[L] /. {Ar[x_, y_, +1] => R[x, y], Ar[x_, y_, -1] => Rinv[x, y]}) //
  βForm,
  Do[
    Do[
      β = β // gm[skel[[s, 1]], skel[[s, k]], skel[[s, 1]]],
      {k, 2, Length[skel[[s]]]}
    ],
    {s, Length[skel]}
  ];
  β // βForm
} // ColumnForm

{Link[9, NonAlternating, 1], {Loop[1, 2, 3, 4], Loop[5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]}

(
  1   h[3]  h[4]  h[5]  h[6]  h[9]  h[11]  h[13]  h[14]  h[17]
  t[1]  0    0    0    - $\frac{-1+T_1}{T_1}$   0    0    0    0    0
  t[2]  0    0    0    0    0    0    0    - $\frac{-1+T_2}{T_2}$   0    0
  t[7]  0    0    0    0    0    0    0    0    - $\frac{-1+T_7}{T_7}$   0
  t[8]  - $\frac{-1+T_8}{T_8}$   0    0    0    0    0    0    0    0    0
  t[10]  0    0    - $\frac{-1+T_{10}}{T_{10}}$   0    0    0    0    0    0    0
  t[12]  0    0    0    0    0    0    0    0    0    - $\frac{-1+T_{12}}{T_{12}}$ 
  t[15]  0    - $\frac{-1+T_{15}}{T_{15}}$   0    0    0    0    0    0    0    0
  t[16]  0    0    0    0    - $\frac{-1+T_{16}}{T_{16}}$   0    0    0    0    0
  t[18]  0    0    0    0    0    0    - $\frac{-1+T_{18}}{T_{18}}$   0    0    0
)

(
   $\frac{T_5+T_1 T_5-T_5^2-2 T_1 T_5^2+2 T_1 T_5^3}{T_1 T_5^4}$           h[1]          h[5]
  t[1]  - $\frac{(-1+T_1)(-1+T_5)(1-T_5+T_5^2)}{T_5^2(1+T_1-T_5-2 T_1 T_5+2 T_1 T_5^2)}$   - $\frac{(-1+T_1)(2-T_5-T_1 T_5^2+2 T_1 T_5^3)}{T_1 T_5^3(1+T_1-T_5-2 T_1 T_5+2 T_1 T_5^2)}$ 
  t[5]  - $\frac{(-1+T_5)(2-T_5-T_1 T_5^2+2 T_1 T_5^3)}{T_5^2(1+T_1-T_5-2 T_1 T_5+2 T_1 T_5^2)}$   - $\frac{(-1+T_5)(1+T_1-T_1 T_5-T_1 T_5^2+2 T_1^2 T_5^2+T_1^2 T_5^3+2 T_1^2 T_5^4-T_1^3 T_5^4-T_1^2 T_5^5+2 T_1^3 T_5^6)}{T_1^2 T_5^5(1+T_1-T_5-2 T_1 T_5+2 T_1 T_5^2)}$ 
)

βMVA[L_Link] := Module[
  {skel, β, s, k},
  skel = Skeleton[L];
  β = Times @@ GD[L] /. {Ar[x_, y_, +1] => R[x, y], Ar[x_, y_, -1] => Rinv[x, y]};
  Do[
    Do[
      β = β // gm[skel[[s, 1]], skel[[s, k]], skel[[s, 1]]],
      {k, 2, Length[skel[[s]]]}
    ],
    {s, Length[skel]}
  ];
  β
]

```

```

Table[{
  skel = Skeleton[L];
  poly =
    (MultivariableAlexander[L][X] /. X[i_] => Tskel[[i,1]] // Numerator // Factor),
  { $\omega$ ,  $\mu$ } = List@@ $\beta$ MVA[L];
  (B[ $\omega$ ,  $\omega * \mu$ ] /. t[s_] => t[s] / (Ts - 1)) //  $\beta$ Form;
  tails = tL[B[ $\omega$ ,  $\mu$ ]]; heads = hL[B[ $\omega$ ,  $\mu$ ]]; vars = (Tu) & /@ tails;
  mat = Transpose[Outer[ $\beta$ Simplify[
    Coefficient[ $\omega * \mu$  /. t[s_] => t[s] / (Ts - 1), h[#1] t[#2]]] &, heads, tails]];
  nums = DeleteCases[Numerator /@ Flatten[mat], 0];
  bas = GroebnerBasis[Append[nums,  $\omega$ ], vars];
  PolynomialReduce[poly, bas, vars]
}, {L, Take[AllLinks[9, NonAlternating], All]}] // ColumnForm

{-2 + T5 + T1 T52 - 2 T1 T53, {{-1 - T1 - 2 T1 T5, 0, 1 + T5, 0}, 0}}
{2 (-1 + T1) (-1 + T5), {{2 - 2 T1 - 2 T5 + 2 T1 T5, 0}}
{(-1 + T1) (-1 + T5), {{1 - T1 - T5 + T1 T5, 0}}
{-1 - T1 T52, {{-1 - T1 T52, 0}}
{(-1 + T1) (-1 + T5) (1 + T52), {{1 - T1 - T5 + T1 T5, 0, 0}, 0}}
{(-1 + T1) (-1 + T5) (1 - T5 + T52), {{1 - T1 - T5 + T1 T5, 0, 0}, 0}}
{-2 + 2 T5 - T1 T5 - T52 + 2 T1 T52 - 2 T1 T53, {{-2 + 2 T5 - T1 T5 - T52 + 2 T1 T52 - 2 T1 T53, 0}}
{(-1 + T1) (-1 + T5) (1 - T5 + T52), {{1 - T5, 0, -1 + 2 T5 - 2 T52 + T53, 0}}
{-1 - T1 + T1 T5 + T52 - T53 - T1 T53, {{1, 0, -1 + T5 - T53, 0}, 0}}
{-1 + T1 + 2 T5 - 3 T1 T5 - 3 T52 + 2 T1 T52 + T53 - T1 T53, {{1 - T1, 0, -1 - T5, 0}, 0}}
{1 - T1 - 2 T5 + T1 T5 + T52 - 2 T1 T52 - T53 + T1 T53, {{-T5 + T1 T5, 0, -1 - T5, 0}, 0}}
{-T1 - T52 + T1 T52 + T53 - T1 T53 - T55, {{-1, 0, -1 + T52 - T53, 0}, 0}}
{-(1 - T1 + T1 T7) (1 - T7 + T1 T7), {{T1 - T12, 0, 1, 0}, 0}}
{-1 + T7 - T1 T7 + T12 T7 - T12 T72, {{-1 + T7 - T1 T7 + T12 T7 - T12 T72, 0}}
{-1 - T1 T72 - T12 T74, {{-1 - T1 T72 - T12 T74, 0}}
{-T1 + T1 T7 - T12 T7 - T1 T72 - T73 + T1 T73 - T1 T74, {{-T1 + T1 T7 - T12 T7 - T1 T72 - T73 + T1 T73 - T1 T74, 0}}
{1 - 2 T1 - T7 + 3 T1 T7 - T12 T7 - 2 T1 T72 + T12 T72, {{-2 T1 + T12, 0, 1 - T1, 0}}
{-(1 + T12 T9) (1 + T1 T92), {{-1 - 2 T1 - T12 - 2 T9 + 3 T1 T9 - 2 T12 T9 - 3 T92 + 5 T1 T92 - 3 T12 T92 + 8 T93 - 9 T1 T93 -
  {-(T12 + T9) (1 + T1 T92), {{
    { $\frac{180}{121} + \frac{7 T_1}{121} - \frac{81 T_9}{121} - \frac{128 T_1 T_9}{121} + \frac{403 T_9^2}{121} - \frac{568}{121} T_1 T_9^2 - \frac{107 T_9^3}{121} + \frac{404}{121} T_1 T_9^3 + \frac{266 T_9^4}{121} - \frac{376}{121} T_1 T_9^4$ ,
    {-T1 T5 + 2 T9 - T1 T9 - T5 T9 + 2 T1 T5 T9 - 2 T92 + T1 T92 + T5 T92 - 2 T1 T5 T92 + T93, {{
      {- $\frac{1}{2} + \frac{T_1}{2} + \frac{T_5}{2} - T_1 T_5 + \frac{T_9}{2}$ , -
      {-T1 T5 + T1 T9 + T5 T9 - T1 T92 - T5 T92 + T93, {{
        {-T1 T5 + T1 T9 + T5 T9 - T1 T92 - T5 T92 - T93 + T1 T93 + T5 T93, {{
          {-1 - T9 + T1 T9 + T5 T9, 0, -1 + T1, 0,
          {1 - T5 - T11 + T1 T52 T11 + T1 T5 T112 - T1 T52 T112, {{
            {1 - T5 - T11 + T1 T52 T11 + T1 T5 T112 - T1 T52 T112, 0}}
          {-T5 + T52 + T1 T11 + 2 T5 T11 - 2 T1 T5 T11 - T52 T11 - T1 T112 + T1 T5 T112, {{
            {-T5 + T52 + T1 T11 + 2 T5 T11 - 2 T1 T5 T11 - 2 T1 T5 T112, 0}}
          {(-1 + T1) (-1 + T5) (-1 + T11), {{
            {-1 + T1 + T5 - T1 T5 + T11 - T1 T11 - T5 T11 + T1 T5 T11, 0}}
          {(-1 + T5) (-T1 + T52) (-1 + T11), {{
            {-T1 + T52, 0, -1, 0}, -1 + T5 + T52}}
          {0, {0}, 0}}
          {(-1 + T1) (-1 + T5)2 (1 + T5 T15), {{
            {-1 + T1 + 2 T5 - 2 T1 T5 - T52 + T1 T52 - T5 T15 + T1 T5 T15 + 2 T52 T15 - 2 T1

```

```

PolynomialReduce[(-1 + T1) (-1 + T5)^2 (1 + T5 T15) T1^3 T5^3 T15^3,
{(-1 + T5) (-1 + T1 + T5 - T1 T5 - T5^2 + T1 T5^2 - T1 T5^3 - T5 T15 + T1 T5 T15 + T5^2 T15 - T1 T5^2 T15 + T1 T5^3 T15),
-T1 (-1 + T5)^2 T5, -T5 (-T5 - T15 + T5 T15 - T1 T5 T15 - T5^2 T15 + T1 T5^2 T15),
(-1 + T5)^2 (-1 + T5 - T5^2 - T5 T15 + T5^2 T15), T1 T5^2 - T15 + 2 T1 T15 - T1^2 T15 + T5 T15 -
2 T1 T5 T15 + 2 T1^2 T5 T15 - T5^2 T15 + 2 T1 T5^2 T15 - T1^2 T5^2 T15 - T1 T5^3 T15 + T1^2 T5^3 T15 +
T1 T5^4 T15 - T1^2 T5^4 T15 - T5 T15^2 + T1 T5 T15^2 - T1^2 T5 T15^2 + T5^2 T15^2 - 2 T1 T5^2 T15^2 + T1^2 T5^2 T15^2 +
T1 T5^3 T15^2 - T1^2 T5^3 T15^2 - T1 T5^4 T15^2 + T1^2 T5^4 T15^2 - T1^3 T15^3 - T1 T5^2 T15^3, -(-1 + T5) T5^3},
{T1, T5, T15}
]

```

```

{ {-1 + 2 T1 - T1^3 + T1 T5 - 2 T1^2 T5 + T1^3 T5 - T1^2 T5^2 + T1^3 T5^2 - T15 + 2 T1 T15 - T1^3 T15 + T1 T5 T15 - 2 T1^2 T5 T15 +
T1^3 T5 T15 - T1^2 T5^2 T15 + T1^3 T5^2 T15 - T15^2 + 2 T1 T15^2 - T1^3 T15^2 + T1 T5 T15^2 - 2 T1^2 T5 T15^2 + T1^3 T5 T15^2 -
T1^2 T5^2 T15^2 + T1^3 T5^2 T15^2 - T15^3 + T1 T15^3 + T1^2 T15^3 - T1^3 T15^3 + T1 T5 T15^3 - T1^2 T5 T15^3 - T1^3 T5^2 T15^3 + T1^3 T5^2 T15^3,
-2 + 2 T1^2 - T1^3 + T1^2 T5^2 - T1^3 T5^2 + T1^2 T5^3 - T1^3 T5^3 + T1 T15 - 2 T1^2 T15 + T1^3 T15 + T1 T15^2 - 2 T1^2 T15^2 +
T1^3 T15^2 + T15^3 - 3 T1^2 T15^3 + 2 T1^3 T15^3 + T15^4 + 2 T1 T15^4 - 3 T1^2 T15^4 + T1^3 T15^4, 0, 0, 0, 0},
1 - 3 T1 + 2 T1^2 + T1^3 - T1^4 - 2 T5 + 3 T1 T5 - T1^2 T5 - 3 T1^3 T5 + 2 T1^4 T5 + 2 T5^2 - T1^2 T5^2 + 2 T1^3 T5^2 - T1^4 T5^2 -
T5^3 + T15 - 3 T1 T15 + 2 T1^2 T15 + T1^3 T15 - T1^4 T15 - T5 T15 + 2 T1 T5 T15 + 2 T1^2 T5 T15 - 6 T1^3 T5 T15 +
3 T1^4 T5 T15 + T1 T5^2 T15 - 4 T1^2 T5^2 T15 + 5 T1^3 T5^2 T15 - 2 T1^4 T5^2 T15 + T15^2 - 3 T1 T15^2 + 2 T1^2 T15^2 +
T1^3 T15^2 - T1^4 T15^2 - T5 T15^2 + 2 T1 T5 T15^2 + 2 T1^2 T5 T15^2 - 6 T1^3 T5 T15^2 + 3 T1^4 T5 T15^2 + T1 T5^2 T15^2 -
4 T1^2 T5^2 T15^2 + 5 T1^3 T5^2 T15^2 - 2 T1^4 T5^2 T15^2 + T15^3 - 2 T1 T15^3 + 2 T1^2 T15^3 - T1^3 T15^3 - T5 T15^3 + T1 T5 T15^3 +
4 T1^2 T5 T15^3 - 7 T1^3 T5 T15^3 + 3 T1^4 T5 T15^3 + T1 T5^2 T15^3 - 4 T1^2 T5^2 T15^3 + 5 T1^3 T5^2 T15^3 - 2 T1^4 T5^2 T15^3 +
T5 T15^4 - T1 T5 T15^4 + 2 T1^2 T5 T15^4 - T1^3 T5 T15^4 - 2 T5^2 T15^4 + T1 T5^2 T15^4 - 2 T1^2 T5^2 T15^4 + T1^3 T5^2 T15^4 + T5^3 T15^4}

```

```

{-T5 + T5^2 + T1 T11 + 2 T5 T11 - 2 T1 T5 T11 - T5^2 T11 - T1 T11^2 + T1 T5 T11^2, (1 - T1 - T5 + T1 T11) / T5,

```

```

( ( 1 h[1] h[5] h[11]
t[1] -(-1+T11)/T5 -1/(T1 T5 T11) 1/T5 T11
t[5] -(-T1+T11+T1 T11)/T5 -(T1-T11-T1 T11+T1 T11^2)/(T1 T5 T11) (T1-T5-T11+T5 T11)/T5 T11 // MatrixForm
t[11] (-1+T5+T1 T5)/T5 (1-T1 T5-T1 T11+T1 T5 T11)/(T1 T5 T11) -(1-T5+T5^2)/T5 T11
)

```

```

{-T5 + T5^2 + T1 T11 + 2 T5 T11 - 2 T1 T5 T11 - T5^2 T11 - T1 T11^2 + T1 T5 T11^2,

```

```

( 1 h[1] h[5] h[11]
t[1] -(-1+T11)/T5 -1/(T1 T5 T11) 1/T5 T11
t[5] -(-T1+T11+T1 T11)/T5 -(T1-T11-T1 T11+T1 T11^2)/(T1 T5 T11) (T1-T5-T11+T5 T11)/T5 T11
t[11] (-1+T5+T1 T5)/T5 (1-T1 T5-T1 T11+T1 T5 T11)/(T1 T5 T11) -(1-T5+T5^2)/T5 T11
)

```

```

PolynomialReduce[
  (-T5 + T5^2 + T1 T11 + 2 T5 T11 - 2 T1 T5 T11 - T5^2 T11 - T1 T11^2 + T1 T5 T11^2) (T1^3 T5^3 T11^3),
  {-T1 + T11 + T1 T11, -1 + T5 + T1 T5, 1 - T1 T5 - T1 T11 + T1 T5 T11,
   -1 + T11, T1 - T11 - T1 T11 + T1 T11^2, 1 - T5 + T5^2},
  {T1, T5, T11}
]
{
  {15 T5^3 - 5 T1 T5^3 + T1^2 T5^3 + 19 T5^4 - 10 T1 T5^4 + 4 T1^2 T5^4 - T1^3 T5^4 - 4 T5^5 + T1 T5^5 + 10 T5^3 T11 - 4 T1 T5^3 T11 +
   T1^2 T5^3 T11 + 9 T5^4 T11 - 6 T1 T5^4 T11 + 3 T1^2 T5^4 T11 - T1^3 T5^4 T11 - 3 T5^5 T11 + T1 T5^5 T11 + 6 T5^3 T11^2 -
   3 T1 T5^3 T11^2 + T1^2 T5^3 T11^2 + 3 T5^4 T11^2 - 3 T1 T5^4 T11^2 + 2 T1^2 T5^4 T11^2 - T1^3 T5^4 T11^2 - 2 T5^5 T11^2 + T1 T5^5 T11^2 +
   3 T5^3 T11^3 - 2 T1 T5^3 T11^3 + T1^2 T5^3 T11^3 - T1 T5^4 T11^3 + 2 T1^2 T5^4 T11^3 - T1^3 T5^4 T11^3 - T5^5 T11^3 + T1 T5^5 T11^3 -
   T1^2 T5^5 T11^3 + T5^4 T11^4 - T1 T5^4 T11^4 + T1^2 T5^4 T11^4 - T1^3 T5^4 T11^4 - T5^5 T11^4 + T1 T5^5 T11^4 - T1^2 T5^5 T11^4 + T1^3 T5^5 T11^4,
   35 T5^3 - 15 T1 T5^3 + 5 T1^2 T5^3 - T1^3 T5^3 - 5 T5^4 + T1 T5^4, 0,
   -35 T5^3 - 30 T5^4 + 10 T5^5 - 20 T5^3 T11 - 11 T5^4 T11 + 6 T5^5 T11 - 10 T5^3 T11^2 - 2 T5^4 T11^2 + 3 T5^5 T11^2 -
   4 T5^3 T11^3 + T5^4 T11^3 + T5^5 T11^3 - T5^3 T11^4 + T5^4 T11^4 + T5^5 T11^4, 0, -15 - 70 T5 - 55 T5^2 + 15 T5^3}, 15 + 55 T5}
}
Det[
  (
    (
      (-T1+T11+T1 T11) / T5,
      (-T1-T11-T1 T11+T1 T11^2) / (T1 T5 T11)
    )
    (
      (-1+T5+T1 T5) / T5,
      (1-T1 T5-T1 T11+T1 T5 T11) / (T1 T5 T11)
    )
  )
] * T1 T5^2 T11 // Expand
T1 T5 - T1^2 T11 - T5 T11 - T1 T5 T11 + T1^2 T5 T11 + T1^2 T11^2
Factor /@ {-T1 + T11 + T1 T11, -1 + T5 + T1 T5,
  1 - T1 T5 - T1 T11 + T1 T5 T11, -1 + T11, T1 - T11 - T1 T11 + T1 T11^2, 1 - T5 + T5^2}
{-T1 + T11 + T1 T11, -1 + T5 + T1 T5,
  1 - T1 T5 - T1 T11 + T1 T5 T11, -1 + T11, T1 - T11 - T1 T11 + T1 T11^2, 1 - T5 + T5^2}
beta /. B[w_, mu_] -> (
  tails = tL[B[w, mu]]; heads = hL[B[w, mu]];
  mat =
    Outer[betaSimplify[Coefficient[w * mu, h[#1] t[#2]]] &, heads, tails] // Transpose;
  Numerator /@ Flatten[mat]
)
{
  (-1 + T1) (-1 + T5)^2 (-1 + T5 - T5^2 - T5 T15 + T5^2 T15), (-1 + T1) (-1 + T5) T15
  (-1 + T1 + T5 - T1 T5 - T5^2 + T1 T5^2 - T1 T5^3 - T5 T15 + T1 T5 T15 + T5^2 T15 - T1 T5^2 T15 + T1 T5^3 T15),
  -(-1 + T1) (-1 + T5)^2 T15,
  (-1 + T5)^2 (-1 + T1 + T5 - T1 T5 - T5^2 + T1 T5^2 - T1 T5^3 - T5 T15 + T1 T5 T15 + T5^2 T15 - T1 T5^2 T15 + T1 T5^3 T15),
  (-1 + T5) (T1 T5^2 - T15 + 2 T1 T15 - T1^2 T15 + T5 T15 - 2 T1 T5 T15 + 2 T1^2 T5 T15 - T5^2 T15 + 2 T1 T5^2 T15 -
   T1^2 T5^2 T15 - T1 T5^3 T15 + T1^2 T5^3 T15 + T1 T5^4 T15 - T1^2 T5^4 T15 - T5 T15^2 + T1 T5 T15^2 - T1^2 T5 T15^2 + T5^2 T15^2 -
   2 T1 T5^2 T15^2 + T1^2 T5^2 T15^2 + T1 T5^3 T15^2 - T1^2 T5^3 T15^2 - T1 T5^4 T15^2 + T1^2 T5^4 T15^2 - T1^2 T15^3 - T1 T5^2 T15^3),
  -(-1 + T5) (-T5 - T15 + T5 T15 - T1 T5 T15 - T5^2 T15 + T1 T5^2 T15), -T1 (-1 + T5)^2 T5 (-1 + T15),
  -T5 (-1 + T15) (-T5 - T15 + T5 T15 - T1 T5 T15 - T5^2 T15 + T1 T5^2 T15), -(-1 + T5) T5^3 (-1 + T15)
}
Numerator[0]
0
PolynomialReduce[12 + x, {0, x}, {x}]
Thread::tdlen : Objects of unequal length in {1}{1, 1} cannot be combined. >>
{{1} {1, 1}, 12}

```

```
{
  t1 = Factor[MultivariableAlexander[#, Program -> "MVA1"][X]],
  t2 = MultivariableAlexander[#, Program -> "MVA2"][X],
  Simplify[t1 / t2], Simplify[(t1 /. {X[1] -> X[2], X[2] -> X[1]}) / t2],
  Simplify[(t1 /. {X[1] -> X[2], X[2] -> X[3], X[3] -> X[1]}) / t2]
} & /@ AllLinks[9, NonAlternating] /. X[i_] -> Xi // MatrixForm
```

Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

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Power::infy : Infinite expression  $\frac{1}{0}$  encountered. >>

General::stop : Further output of Power::infy will be suppressed during this calculation. >>

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. >>

General::stop : Further output of Infinity::indet will be suppressed during this calculation. >>

$-2 + X_1 + X_1^2 X_2 - 2 X_1^3 X_2$	$-\frac{2 - X_2 - X_1 X_2^2 + 2 X_1}{\sqrt{X_1} X_2^{3/2}}$
$-2 (-1 + X_1) (-1 + X_2)$	$\frac{2 (-1 + X_1) (-1 + X_2)}{\sqrt{X_1} \sqrt{X_2}}$
$- (-1 + X_1) (-1 + X_2)$	$\frac{(-1 + X_1) (-1 + X_2)}{\sqrt{X_1} \sqrt{X_2}}$
$-1 - X_1^5 X_2$	$\frac{-1 - X_1 X_2^5}{\sqrt{X_1} X_2^{5/2}}$
$- (-1 + X_1) (1 + X_1^2) (-1 + X_2)$	$\frac{(-1 + X_1) (-1 + X_2) (1)}{\sqrt{X_1} X_2^{3/2}}$
$- (-1 + X_1) (1 - X_1 + X_1^2) (-1 + X_2)$	$\frac{(-1 + X_1) (-1 + X_2) (1 - X_1 + X_1^2)}{\sqrt{X_1} X_2^{3/2}}$
$-2 + 2 X_1 - X_1^2 - X_1 X_2 + 2 X_1^2 X_2 - 2 X_1^3 X_2$	$-\frac{2 - 2 X_2 + X_1 X_2 + X_2^2 - 2 X_1 X_2}{\sqrt{X_1} X_2^{3/2}}$
$- (-1 + X_1) (1 - X_1 + X_1^2) (-1 + X_2)$	$\frac{(-1 + X_1) (-1 + X_2) (1 - X_1 + X_1^2)}{\sqrt{X_1} X_2^{3/2}}$
$-1 + X_1^2 - X_1^3 - X_2 + X_1 X_2 - X_1^3 X_2$	$-\frac{1 + X_1 - X_1 X_2 - X_2^2 + X_2^3}{\sqrt{X_1} X_2^{3/2}}$
$-1 + 2 X_1 - 3 X_1^2 + X_1^3 + X_2 - 3 X_1 X_2 + 2 X_1^2 X_2 - X_1^3 X_2$	$-\frac{1 - X_1 - 2 X_2 + 3 X_1 X_2 + 3 X_2^2 - 2 X_1 X_2}{\sqrt{X_1} X_2^{3/2}}$
$-1 + 2 X_1 - X_1^2 + X_1^3 + X_2 - X_1 X_2 + 2 X_1^2 X_2 - X_1^3 X_2$	$\frac{1 - X_1 - 2 X_2 + X_1 X_2 + X_2^2 - 2 X_1 X_2}{\sqrt{X_1} X_2^{3/2}}$
$-X_1^2 + X_1^3 - X_1^5 - X_2 + X_1^2 X_2 - X_1^3 X_2$	$\frac{-X_1 - X_2^2 + X_1 X_2^2 + X_2^3 - X_1 X_2}{\sqrt{X_1} X_2^{5/2}}$
$- (1 - X_1 + X_1 X_2) (1 - X_2 + X_1 X_2)$	$-\frac{(1 - X_1 + X_1 X_2) (1 - X_2 + X_1 X_2)}{X_1 X_2}$
$-1 + X_2 - X_1 X_2 + X_1^2 X_2 - X_1^2 X_2^2$	$-\frac{1 - X_2 + X_1 X_2 - X_1^2 X_2 + X_1^2 X_2^2}{X_1 X_2}$
$-1 - X_1^2 X_2 - X_1^4 X_2^2$	$-\frac{1 + X_1 X_2^2 + X_1^2 X_2^2}{X_1 X_2^2}$

$-X_1^3 - X_2 + X_1 X_2 - X_1^2 X_2 + X_1^3 X_2 - X_1^4 X_2 - X_1 X_2^2$	$-\frac{X_1 - X_1 X_2 + X_1^2 X_2 + X_1 X_2^2 + X_2^3 -}{X_1 X_2^2}$
$-1 + X_1 + 2 X_2 - 3 X_1 X_2 + 2 X_1^2 X_2 + X_1 X_2^2 - X_1^2 X_2^2$	$\frac{1 - 2 X_1 - X_2 + 3 X_1 X_2 - X_1^2 X_2 - 2}{X_1 X_2}$
$- (1 + X_1^2 X_2) (1 + X_1 X_2^2)$	$-\frac{(1 + X_1^2 X_2) (1 + X_1 X_2^2)}{X_1^{3/2} X_2^{3/2}}$
$- (X_1^2 + X_2) (1 + X_1 X_2^2)$	$-\frac{(X_1^2 + X_2) (1 + X_1 X_2^2)}{X_1^{3/2} X_2^{3/2}}$
$2 X_1 - 2 X_1^2 + X_1^3 - X_1 X_2 + X_1^2 X_2 - X_1 X_3 + X_1^2 X_3 - X_2 X_3 + 2 X_1 X_2 X_3 - 2 X_1^2 X_2 X_3$	$\frac{-X_1 X_2 + 2 X_3 - X_1 X_3 - X_2 X_3 + 2 X_1 X_2 X_3 - 2 X_1^2 X_2 X_3 +}{\sqrt{X_1} \sqrt{X_2} X_3^{3/2}}$
$X_1^3 + X_1 X_2 - X_1^2 X_2 + X_1 X_3 - X_1^2 X_3 - X_2 X_3$	$\frac{-X_1 X_2 + X_1 X_3 + X_2 X_3 - X_1 X_3^2 - X_2^2 X_3}{\sqrt{X_1} \sqrt{X_2} X_3^{3/2}}$
$X_1^3 + X_2 - X_1 X_2 + X_1^2 X_2 - X_1^3 X_2 + X_3 - X_1 X_3 + X_1^2 X_3 - X_1^3 X_3 - X_2 X_3$	$\frac{-X_1 - X_2 + X_1 X_2 + X_1 X_3 + X_2 X_3 - X_1 X_3^2 - X_2^2 X_3}{\sqrt{X_1} \sqrt{X_2} X_3^{3/2}}$
$1 - X_1 - X_2 + X_1^2 X_2 X_3 + X_1 X_2^2 X_3 - X_1^2 X_2^2 X_3$	$\frac{1 - X_2 - X_3 + X_1 X_2^2 X_3 + X_1 X_2 X_3^2}{\sqrt{X_1} X_2 X_3}$
$X_2 - 2 X_1 X_2 - X_2^2 + X_1 X_2^2 - X_1 X_3 + X_1^2 X_3 + 2 X_1 X_2 X_3 - X_1^2 X_2 X_3$	$\frac{-X_2 + X_2^2 + X_1 X_3 + 2 X_2 X_3 - 2 X_1 X_2 X_3 - X_1^2 X_2 X_3}{\sqrt{X_1} X_2 X_3}$
$- (-1 + X_1) (-1 + X_2) (-1 + X_3)$	$\frac{(-1 + X_1) (-1 + X_2) (-1 + X_3)}{\sqrt{X_1} \sqrt{X_2} \sqrt{X_3}}$
$(-1 + X_1) (-1 + X_2) (X_1^2 - X_3)$	$\frac{(-1 + X_2) (-1 + X_3) (-X_1^2)}{\sqrt{X_1} \sqrt{X_2} X_3^{3/2}}$
$0$	$0$
$- (-1 + X_1)^2 (1 + X_1 X_2) (-1 + X_3)$	$\frac{(-1 + X_1) (-1 + X_3)^2 (1 + X_1 X_2)}{\sqrt{X_1} \sqrt{X_2} X_3^{3/2}}$