Pensieve Header: Computing the Alexander polynomial in β-calculus, a minimal approach.

```
<< KnotTheory`
GC[K_] := GC @@ (
   PD[K] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]],
      Ar[l, i, +1], Ar[j, i, -1]
     ]
  )
```

```
Loading KnotTheory` version of August 22, 2010, 13:36:57.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Alexander[K = Knot[9, 42]][X]
```

$$-1-\frac{1}{X^2}+\frac{2}{X}+2X-X^2$$

```
βSimplify = Factor;
SetAttributes[βCollect, Listable];
βCollect[B[ω_, μ_]] := B[
   βSimplify[ω],
   Collect[μ, _h, Collect[#, _t, βSimplify] &]
  ];
(* "L" for "Labels" *)
hL[β_] := Union[Cases[β, h[s_] :> s, Infinity]];
tL[β_] := Union[Cases[β, t[s_] | c_s_ :> s, Infinity]];
dL[β_] := Union[hL[β], tL[β]];
SetAttributes[βForm, Listable];
βForm[B[ω_, μ_]] := Module[
   {tails, heads, mat},
   tails = tL[B[ω, μ]]; heads = hL[B[ω, μ]];
   mat = Outer[βSimplify[Coefficient[μ, h[#1] t[#2]]] &, heads, tails];
   PrependTo[mat, t /@ tails];
   mat = Prepend[Transpose[mat], Prepend[h /@ heads, ω]];
   MatrixForm[mat]
  ];
```

```
R[x_, y_] := B[1, (E^c_x - 1) / c_x * t[x] h[y]];
Rinv[x_, y_] := B[1, (E^(-c_x) - 1) / c_x * t[x] h[y]];
tm[x_, y_, z_][β_] := β /. {t[x] → t[z], t[y] → t[z], c_x → c_z, c_y → c_z};
hm[x_, y_, z_][B[ω_, μ_]] := Module[
   {γx = D[μ, h[x]], γy = D[μ, h[y]], M = μ /. h[x] | h[y] → 0},
   B[ω, M + h[z] (γx + γy + (γx /. t[i_] :> c_i) γy)] // βCollect
  ];
swap[x_, y_][B[ω_, μ_]] := Module[
   {α, β, γ, δ, ϵ},
   α = Coefficient[μ, h[x] t[y]];
   β = D[μ, t[y]] /. h[x] → 0;
   γ = D[μ, h[x]] /. t[y] → 0;
   δ = μ /. h[x] | t[y] → 0;
   ϵ = 1 + c_y α;
   B[ω * ϵ, Plus[
      α (1 + (γ /. t[i_] :> c_i) / ϵ) h[x] t[y],
      β (1 + (γ /. t[i_] :> c_i) / ϵ) t[y],
      γ / ϵ h[x],
      δ - c_y / ϵ γ * β
     ]] // βCollect
  ];
gm[x_, y_, z_][β_] := β // swap[y, x] // hm[x, y, z] // tm[x, y, z];
B /: B[ω1_, μ1_] B[ω2_, μ2_] := B[ω1 * ω2, μ1 + μ2];

{
  β = B[ω[c_1, c_2, c_3, c_4], Sum[α_i[c_1, c_2, c_3, c_4] t[i] h[1], {i, 4}]],
  β // tm[1, 2, 1],
  t1 = β // tm[1, 2, 1] // tm[1, 3, 1],
  t2 = β // tm[2, 3, 28] // tm[1, 28, 1],
  t1 == t2
 } // βForm
```

$$\left\{\begin{pmatrix} \omega[c_1, c_2, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_2, c_3, c_4] \\ t[2] & \alpha_2[c_1, c_2, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_2, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_2, c_3, c_4] \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega[c_1, c_1, c_3, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_3, c_4] + \alpha_2[c_1, c_1, c_3, c_4] \\ t[3] & \alpha_3[c_1, c_1, c_3, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_3, c_4] \end{pmatrix},$$

$$\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega[c_1, c_1, c_1, c_4] & h[1] \\ t[1] & \alpha_1[c_1, c_1, c_1, c_4] + \alpha_2[c_1, c_1, c_1, c_4] + \alpha_3[c_1, c_1, c_1, c_4] \\ t[4] & \alpha_4[c_1, c_1, c_1, c_4] \end{pmatrix}, \beta\text{Form}[\text{True}]\right\}$$

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 4}]],
  β // hm[1, 2, 1],
  t1 = β // hm[1, 2, 1] // hm[1, 3, 1],
  t2 = β // hm[2, 3, 28] // hm[1, 28, 1],
  t1 == t2
 } // βForm // ColumnForm
```

ω	h[1]	h[2]	h[3]	h[4]
t[1]	α_{11}	α_{12}	α_{13}	α_{14}
t[2]	α_{21}	α_{22}	α_{23}	α_{24}

ω	h[1]	h[3]	h[4]
t[1]	$\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21}$	α_{13}	α_{14}
t[2]	$\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}$	α_{23}	α_{24}

ω	h[1]
t[1]	$\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+\alpha_{13}+c_1\,\alpha_{11}\,\alpha_{13}+c_1\,\alpha_{12}\,\alpha_{13}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{13}+c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{13}\,\alpha_{21}+c_1\,c_2\,\alpha_{12}$ (
t[2]	$\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}+\alpha_{23}+c_1\,\alpha_{11}\,\alpha_{23}+c_1\,\alpha_{12}\,\alpha_{23}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{23}+c_2\,\alpha_{21}\,\alpha_{23}+c_1\,c_2\,\alpha_{12}$ (

ω	h[1]
t[1]	$\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+\alpha_{13}+c_1\,\alpha_{11}\,\alpha_{13}+c_1\,\alpha_{12}\,\alpha_{13}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{13}+c_2\,\alpha_{12}\,\alpha_{21}+c_2\,\alpha_{13}\,\alpha_{21}+c_1\,c_2\,\alpha_{12}$ (
t[2]	$\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}+\alpha_{23}+c_1\,\alpha_{11}\,\alpha_{23}+c_1\,\alpha_{12}\,\alpha_{23}+c_1^2\,\alpha_{11}\,\alpha_{12}\,\alpha_{23}+c_2\,\alpha_{21}\,\alpha_{23}+c_1\,c_2\,\alpha_{12}$ (

βForm[True]

```
Clear[β];
{β1 = B[ω, h[1] t[1] α + h[2] t[1] β + h[1] t[2] γ + h[2] t[2] δ],
  β1 // swap[1, 1]
 } // βForm
```

{

ω	h[1]	h[2]
t[1]	α	β
t[2]	γ	δ

,

$\omega\,(1+\alpha\,c_1)$	h[1]	h[2]
t[1]	$\frac{\alpha\,(1+\alpha\,c_1+\gamma\,c_2)}{1+\alpha\,c_1}$	$\frac{\beta\,(1+\alpha\,c_1+\gamma\,c_2)}{1+\alpha\,c_1}$
t[2]	$\frac{\gamma}{1+\alpha\,c_1}$	$\frac{\delta+\beta\,\gamma\,c_1+\alpha\,\delta\,c_1}{1+\alpha\,c_1}$

}

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 2}, {j, 3}]],
  β // hm[1, 2, 1],
  t1 = β // hm[1, 2, 1] // swap[1, 1],
  t2 = β // swap[1, 1] // swap[2, 1] // hm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega & h[1] & h[3] \\ t[1] & \alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21} & \alpha_{13} \\ t[2] & \alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22} & \alpha_{23} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}) & h[1] & \\ t[1] & \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,(\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21})\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} & \\ t[2] & \frac{\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} & -c_1\,\alpha_1 \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}) & h[1] & \\ t[1] & \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,(\alpha_{11}+\alpha_{12}+c_1\,\alpha_{11}\,\alpha_{12}+c_2\,\alpha_{12}\,\alpha_{21})\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} & \\ t[2] & \frac{\alpha_{21}+\alpha_{22}+c_1\,\alpha_{11}\,\alpha_{22}+c_2\,\alpha_{21}\,\alpha_{22}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{12}+c_1^2\,\alpha_{11}\,\alpha_{12}+c_1\,c_2\,\alpha_{12}\,\alpha_{21}} & -c_1\,\alpha_1 \end{pmatrix}$$

```
βForm[True]
βForm[True]
```

```
{
  β = B[ω, Sum[α10 i+j t[i] h[j], {i, 3}, {j, 2}]],
  t1 = β // tm[1, 2, 1] // swap[1, 1],
  t2 = β // swap[1, 2] // swap[1, 1] // tm[1, 2, 1],
  First[t1] == First[t2],
  Last[t1] == Last[t2] // Simplify
 } // βForm // ColumnForm
```

$$\begin{pmatrix} \omega & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \\ t[3] & \alpha_{31} & \alpha_{32} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{(\alpha_{11}+\alpha_{21})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{(\alpha_{12}+\alpha_{22})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{-c_1\,\alpha_{12}\,\alpha_{31}-c_1\,\alpha_{22}\,\alpha_{31}+\alpha_{32}+c_1\,\alpha_{11}\,\alpha_{32}+c_1\,\alpha_{21}\,\alpha_{32}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \end{pmatrix}$$

$$\begin{pmatrix} \omega\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}) & h[1] & h[2] \\ t[1] & \frac{(\alpha_{11}+\alpha_{21})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{(\alpha_{12}+\alpha_{22})\,(1+c_1\,\alpha_{11}+c_1\,\alpha_{21}+c_3\,\alpha_{31})}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \\ t[3] & \frac{\alpha_{31}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} & \frac{-c_1\,\alpha_{12}\,\alpha_{31}-c_1\,\alpha_{22}\,\alpha_{31}+\alpha_{32}+c_1\,\alpha_{11}\,\alpha_{32}+c_1\,\alpha_{21}\,\alpha_{32}}{1+c_1\,\alpha_{11}+c_1\,\alpha_{21}} \end{pmatrix}$$

```
βForm[True]
βForm[True]
```

```
{
   β = B[ω, Sum[α10 i+j t[i] h[j], {i, 4}, {j, 4}]],
   t1 = β // gm[1, 2, 1] // gm[1, 3, 1],
   t2 = β // gm[2, 3, 2] // gm[1, 2, 1],
   First[t1] == First[t2],
   Last[t1] == Last[t2] // Simplify
  } // βForm // ColumnForm
```

A very large output was generated. Here is a sample of it:

$$\begin{pmatrix} \omega & h[1] & h[2] & h[3] & h[4] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ t[4] & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1 + c_1\,\alpha_{12} + c_1\,\alpha_{13} + c_1^2\,\alpha_{12}\,\alpha_{13} + c_1\,\alpha_{23} + c_1^2\,\alpha_{12}\,\alpha_{23} + c_1^2\,\alpha_{13}\,\alpha_{32} + c_1\,c_4\,\alpha_{13}\,\alpha_{42}\right) & h \\ t[1] & \frac{\alpha_{11}+\alpha_{12}+\ll 644\gg + c_1\,c_4^3\,\alpha_{}}{1+c_1\,\alpha_{12}+c_1\,\alpha_{13}+c_1^2\,\alpha_{12}\,\alpha_{13}+c_1\,\alpha_{}} \\ t[4] & \end{pmatrix}$$

$$\begin{pmatrix} \omega\left(1 + c_1\,\alpha_{12} + c_1\,\alpha_{13} + c_1^2\,\alpha_{12}\,\alpha_{13} + c_1\,\alpha_{23} + c_1^2\,\alpha_{12}\,\alpha_{23} + c_1^2\,\alpha_{13}\,\alpha_{32} + c_1\,c_4\,\alpha_{13}\,\alpha_{42}\right) & h \\ t[1] & \frac{\alpha_{11}+\alpha_{12}+2\,c_1\,\alpha_{11}\,\alpha_{12}+\ll 643\gg + c}{1+c_1\,\alpha_{12}+c_1\,\alpha_{13}+c_1^2\,\alpha_{12}\,\alpha_{13}+c_1\,\alpha} \\ t[4] & \end{pmatrix}$$

```
βForm[True]
βForm[True]
```

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```
(β = Times @@ GC[K] /. {Ar[x_, y_, +1] :> R[x, y], Ar[x_, y_, -1] :> Rinv[x, y]}) //
 βForm
```

$$\begin{pmatrix} 1 & h[1] & h[3] & h[5] & h[6] & h[9] & h[12] & h[14] & h[16] & h[18] \\ t[2] & 0 & 0 & 0 & 0 & \frac{-1+e^{c_2}}{c_2} & 0 & 0 & 0 & 0 \\ t[4] & -\frac{e^{-c_4}\,(-1+e^{c_4})}{c_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[7] & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7} & 0 & 0 \\ t[8] & 0 & \frac{-1+e^{c_8}}{c_8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t[10] & 0 & 0 & -\frac{e^{-c_{10}}\,(-1+e^{c_{10}})}{c_{10}} & 0 & 0 & 0 & 0 & 0 & 0 \\ t[11] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{11}}}{c_{11}} & 0 \\ t[13] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{13}}}{c_{13}} \\ t[15] & 0 & 0 & 0 & -\frac{e^{-c_{15}}\,(-1+e^{c_{15}})}{c_{15}} & 0 & 0 & 0 & 0 & 0 \\ t[17] & 0 & 0 & 0 & 0 & 0 & \frac{-1+e^{c_{17}}}{c_{17}} & 0 & 0 & 0 \end{pmatrix}$$

```
Table[
   β = β // gm[1, k, 1],
   {k, 2, 2 Crossings[K]}
  ] // βForm // ColumnForm
```

1	h[1]	h[3]	h[5]	h[6]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	0	0	0	0	$\frac{-1+e^{c_1}}{c_1}$	0	0	0	0
t[4]	$-\frac{e^{-c_4}\,(-1+e^{c_4})}{c_4}$	0	0	0	0	0	0	0	0
t[7]	0	0	0	0	0	0	$-\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7}$	0	0
t[8]	0	$\frac{-1+e^{c_8}}{c_8}$	0	0	0	0	0	0	0
t[10]	0	0	$-\frac{e^{-c_{10}}\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	0	0	0	$-\frac{e^{-c_{15}}\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0	0
t[17]	0	0	0	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[5]	h[6]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	0	0	0	$\frac{e^{c_8}\,(-1+e^{c_1})}{c_1}$	0	0	0	0
t[4]	$-\frac{e^{-c_4}\,(-1+e^{c_4})}{c_4}$	0	0	0	0	0	0	0
t[7]	0	0	0	0	0	$-\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7}$	0	0
t[8]	$\frac{e^{-c_4}\,(-1+e^{c_8})}{c_8}$	0	0	$-\frac{(-1+e^{c_1})\,(-1+e^{c_8})}{c_8}$	0	0	0	0
t[10]	0	$-\frac{e^{-c_{10}}\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	0	0	$-\frac{e^{-c_{15}}\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0	0
t[17]	0	0	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[5]	h[6]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	$-\frac{e^{-c_1}\,(-1+e^{c_1})}{c_1}$	0	0	$\frac{e^{c_8}\,(-1+e^{c_1})}{c_1}$	0	0	0	0
t[7]	0	0	0	0	0	$-\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7}$	0	0
t[8]	$\frac{e^{-c_1}\,(-1+e^{c_8})}{c_8}$	0	0	$-\frac{(-1+e^{c_1})\,(-1+e^{c_8})}{c_8}$	0	0	0	0
t[10]	0	$-\frac{e^{-c_{10}}\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	0	0	$-\frac{e^{-c_{15}}\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0	0
t[17]	0	0	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[6]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	$-\frac{e^{-c_1-c_{10}}\,(-1+e^{c_1})}{c_1}$	0	$\frac{e^{c_8-c_{10}}\,(-1+e^{c_1})}{c_1}$	0	0	0	0
t[7]	0	0	0	0	$-\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7}$	0	0
t[8]	$\frac{e^{-c_1}\,(-1+e^{c_8})}{c_8}$	0	$-\frac{(-1+e^{c_1})\,(-1+e^{c_8})}{c_8}$	0	0	0	0
t[10]	$-\frac{e^{-c_1-c_{10}}\,(-1+e^{c_1}+e^{c_8})\,(-1+e^{c_{10}})}{c_{10}}$	0	$\frac{e^{c_8-c_{10}}\,(-1+e^{c_1})\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0
t[11]	0	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	0	$-\frac{e^{-c_{15}}\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0	0
t[17]	0	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	$-\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+e^{c_1})}{c_1}$	$\frac{e^{c_8-c_{10}-c_{15}}\,(-1+e^{c_1})}{c_1}$	0	0	0	0
t[7]	0	0	0	$-\frac{e^{-c_7}\,(-1+e^{c_7})}{c_7}$	0	0
t[8]	$\frac{e^{-c_1}\,(-1+e^{c_8})}{c_8}$	$-\frac{(-1+e^{c_1})\,(-1+e^{c_8})}{c_8}$	0	0	0	0
t[10]	$-\frac{e^{-c_1-c_{10}}\,(-1+e^{c_1}+e^{c_8})\,(-1+e^{c_{10}})}{c_{10}}$	$\frac{e^{c_8-c_{10}}\,(-1+e^{c_1})\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0
t[11]	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	$-\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+e^{c_1}+e^{c_8})\,(-1+e^{c_{15}})}{c_{15}}$	$\frac{e^{c_8-c_{10}-c_{15}}\,(-1+e^{c_1})\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0
t[17]	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	$-\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+e^{c_1})}{c_1}$	$\frac{e^{c_8-c_{10}-c_{15}}\,(-1+e^{c_1})}{c_1}$	0	$-\frac{e^{-c_1}\,(-1+e^{c_1})}{c_1}$	0	0
t[8]	$\frac{e^{-c_1}\,(-1+e^{c_8})}{c_8}$	$-\frac{(-1+e^{c_1})\,(-1+e^{c_8})}{c_8}$	0	0	0	0
t[10]	$-\frac{e^{-c_1-c_{10}}\,(-1+e^{c_1}+e^{c_8})\,(-1+e^{c_{10}})}{c_{10}}$	$\frac{e^{c_8-c_{10}}\,(-1+e^{c_1})\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0
t[11]	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	$-\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+e^{c_1}+e^{c_8})\,(-1+e^{c_{15}})}{c_{15}}$	$\frac{e^{c_8-c_{10}-c_{15}}\,(-1+e^{c_1})\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0
t[17]	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

1	h[1]	h[9]	h[12]	h[14]	h[16]	h[18]
t[1]	$\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+e^{c_1})\,(-1+e^{c_{10}+c_{15}})}{c_1}$	$-\frac{e^{-c_{10}-c_{15}}\,(-1+e^{c_1})\,(-e^{c_1-c_{10}+c_{15}}+e^{c_1+c_{10}+c_{15}})}{c_1}$	0	$-\frac{e^{-c_1}\,(-1+e^{c_1})}{c_1}$	0	0
t[10]	$-\frac{e^{-c_1-c_{10}}\,(-1+2\,e^{c_1})\,(-1+e^{c_{10}})}{c_{10}}$	$\frac{e^{c_1-c_{10}}\,(-1+e^{c_1})\,(-1+e^{c_{10}})}{c_{10}}$	0	0	0	0
t[11]	0	0	0	0	$\frac{-1+e^{c_{11}}}{c_{11}}$	0
t[13]	0	0	0	0	0	$\frac{-1+e^{c_{13}}}{c_{13}}$
t[15]	$-\frac{e^{-c_1-c_{10}-c_{15}}\,(-1+2\,e^{c_1})\,(-1+e^{c_{15}})}{c_{15}}$	$\frac{e^{c_1-c_{10}-c_{15}}\,(-1+e^{c_1})\,(-1+e^{c_{15}})}{c_{15}}$	0	0	0	0
t[17]	0	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0	0	0

$$\left(-e^{c_1-c_{10}-c_{15}}\left(1-e^{c_1}-2e^{c_{10}+c_{15}}+e^{c_1+c_{10}+c_{15}}\right)\right)$$

	h[1]	h[12]
t[1]	$-\frac{e^{-c_1-c_{10}-c_{15}}\left(-1+e^{c_1}\right)\left(e^{2c_1}-e^{c_{10}+c_{15}+c_1+c_{10}+c_{15}}-e^{2c_1+c_{10}+c_{15}+e^{2c_{10}+2c_{15}}}\right)}{\left(1-e^{c_1}-2e^{c_{10}+c_{15}+e^{c_1+c_{10}+c_{15}}}\right)c_1}$	0
t[10]	$-\frac{e^{-c_1+c_{15}}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{10}}\right)}{\left(1-e^{c_1}-2e^{c_{10}+c_{15}+e^{c_1+c_{10}+c_{15}}}\right)c_{10}}$	0
t[11]	0	0
t[13]	0	0
t[15]	$-\frac{e^{-c_1}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_{10}+c_{15}+e^{c_1+c_{10}+c_{15}}}\right)c_{15}}$	0
t[17]	0	$\frac{-1+e^{c_{17}}}{c_{17}}$

$$\left(-e^{-c_{15}}\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)\right)$$

	h[1]	h[12]	h[14]
t[1]	$-\frac{e^{-c_1-c_{15}}\left(-1+e^{c_1}\right)\left(-1+e^{c_{15}}\right)\left(-e^{c_1}+e^{c_{15}}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_1}$	0	$-\frac{e^{-c_1+c_{15}}\left(-1+e^{c_1}\right)\left(1$
t[11]	0	0	0
t[13]	0	0	0
t[15]	$-\frac{e^{-c_1}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{15}}$	0	$-\frac{e^{-c_1}\left(-1+e^{c_1}\right)^2$
t[17]	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0

$$\left(-e^{-c_{15}}\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)\right)$$

	h[1]	h[12]	h[14]
t[1]	$-\frac{e^{-c_1-c_{15}}\left(-1+e^{c_1}\right)\left(-1+e^{c_{15}}\right)\left(-e^{c_1}+e^{c_{15}}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_1}$	0	$-\frac{e^{-c_1+c_{15}}\left(-1+e^{c_1}\right)\left(1$
t[13]	0	0	0
t[15]	$-\frac{e^{-c_1}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{15}}$	0	$-\frac{e^{-c_1}\left(-1+e^{c_1}\right)^2$
t[17]	0	$\frac{-1+e^{c_{17}}}{c_{17}}$	0

$$\left(-e^{-c_{15}}\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)\right)$$

	h[1]	h[14]
t[1]	$-\frac{e^{-c_1-c_{15}+c_{17}}\left(-1+e^{c_1}\right)\left(-1+e^{c_{15}}\right)\left(-e^{c_1}+e^{c_{15}}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_1}$	$-\frac{e^{-c_1+c_{15}+c_{17}}\left(-1+e^{c_1}\right)\left(1-\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$
t[13]	0	0
t[15]	$-\frac{e^{-c_1}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{15}}$	$-\frac{e^{-c_1}\left(-1+e^{c_1}\right)^2\left(-1\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$
t[17]	$\frac{e^{-c_1}\left(1-2e^{c_1}-e^{c_{15}}+3e^{c_1+c_{15}}-3e^{2c_1+c_{15}}+e^{3c_1+c_{15}}\right)\left(-1+e^{c_{17}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{17}}$	$\frac{e^{-c_1+c_{15}}\left(-1+e^{c_1}\right)\left(1-3e^{c_1}+e^{2}\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$

$$\left(-e^{-c_{15}}\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)\right)$$

	h[1]	h[14]
t[1]	$-\frac{e^{-c_1-c_{15}+c_{17}}\left(-1+e^{c_1}\right)\left(-1+e^{c_{15}}\right)\left(-e^{c_1}+e^{c_{15}}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_1}$	$-\frac{e^{-c_1+c_{15}+c_{17}}\left(-1+e^{c_1}\right)\left(1-\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$
t[15]	$-\frac{e^{-c_1}\left(1-3e^{c_1}+e^{2c_1}\right)\left(-1+e^{c_{15}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{15}}$	$-\frac{e^{-c_1}\left(-1+e^{c_1}\right)^2\left(-1\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$
t[17]	$\frac{e^{-c_1}\left(1-2e^{c_1}-e^{c_{15}}+3e^{c_1+c_{15}}-3e^{2c_1+c_{15}}+e^{3c_1+c_{15}}\right)\left(-1+e^{c_{17}}\right)}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1+c_{15}}\right)c_{17}}$	$\frac{e^{-c_1+c_{15}}\left(-1+e^{c_1}\right)\left(1-3e^{c_1}+e^{2}\right.}{\left(1-e^{c_1}-2e^{c_1+c_{15}}+e^{2c_1}\right.}$

$$\left(e^{-c_1-c_{15}}\left(-e^{c_1}+e^{2c_1}+2e^{2c_1+c_{15}}-e^{3c_1+c_{15}}-e^{c_{15}+c_{17}}+4e^{c_1+c_{15}+c_{17}}-4e^{2c_1+c_{15}+c_{17}}+e^{3c_1+c_{15}+c_{17}}\right)\right)$$

t[1]	$\frac{e^{-c_1-c_{15}+c_{17}}\left(-1+e^{c}\right.}{\left(-e^{c_1}\right.}$
t[15]	$\frac{}{\left(-e^{c_1}\right.}$
t[17]	$-\frac{}{\left(-e^{c}\right.}$

$$\begin{pmatrix} e^{-2c_1}\left(-e^{c_1}+e^{2c_1}+2e^{3c_1}-e^{4c_1}-e^{c_1+c_{17}}+4e^{2c_1+c_{17}}-4e^{3c_1+c_{17}}+e^{4c_1+c_{17}}\right) & \frac{e^{-2c_1}(-1+e^{c_1})\left(e^{c_1}-3e^{2c_1}+e^{3c_1}+e^{c_{17}}+e^{2}\right.}{\left(-1+e^{c_1}+2e^{2c_1}-e^{3c_1}-e^{c_1}\right.} \\ t[1] & -\frac{e^{-c_1}\left(1-3e^{c_1}+3e\right.}{\left(-1+e^{c_1}+2e^{2c_1}-e^{3c_1}-e^{c_1}\right.} \\ t[17] & h[1] \\ \left(-e^{-c_1}\left(1-e^{c_1}-2e^{2c_1}+e^{3c_1}-e^{c_{17}}+3e^{c_1+c_{17}}-3e^{2c_1+c_{17}}+e^{3c_1+c_{17}}\right)\right. & -\frac{e^{-c_1}(-1+e^{c_1})\left(e^{c_1}-3e^{2c_1}+e^{3c_1}+e^{c_{17}}-e^{2c_{17}}-e^{c_1+c_{17}}\right.}{\left(1-e^{c_1}-2e^{2c_1}+e^{3c_1}-e^{c_{17}}+3e^{c_1+c_{17}}\right.} \\ t[1] & \frac{(-2+e^{c_1})(-1+2e^{c_1})(}{\left(1-e^{c_1}-2e^{2c_1}+e^{3c_1}-e^{c_{17}}+3e^{c_1+c_{17}}-3\right.} \\ t[17] & \end{pmatrix}$$

$$\begin{pmatrix} -e^{-c_1}\left(1-2e^{c_1}+e^{2c_1}-2e^{3c_1}+e^{4c_1}\right) & h[1] & h[18] \\ t[1] & 0 & \frac{-1+e^{c_1}}{c_1} \end{pmatrix}$$

$$\begin{pmatrix} -1+2e^{c_1}-e^{2c_1}+2e^{3c_1}-e^{4c_1} & h[1] \\ t[1] & \frac{-1+e^{c_1}}{c_1} \end{pmatrix}$$