Pensieve Header: The w equations in the β-calculus.

β is to remind of “B picture”, though it is “wheeled”. Also, in faux German, β is ß is SS, for “semi-symmetrized”.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-01"];
<< betaCalculus.m
```

Capping a Strand

```
{R[1, 2], R[1, 2] // hη[1], R[1, 2] // hη[2]} // βForm
```

$$\left\{\begin{pmatrix} W[1] & h[2] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} \end{pmatrix}, \begin{pmatrix} W[1] \end{pmatrix}\right\}$$

```
{ρ = W[E^c[1]] + R[1, 2], ρ // hη[1], ρ // hη[2], ρ // dη[1], ρ // dη[2]} // βForm
```

$$\left\{\begin{pmatrix} W\left[e^{c[1]}\right] & h[2] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} \end{pmatrix}, \begin{pmatrix} W\left[e^{c[1]}\right] & h[2] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} \end{pmatrix}, \begin{pmatrix} W\left[e^{c[1]}\right] \\ t[1] \end{pmatrix}, \begin{pmatrix} W[1] \end{pmatrix}, \begin{pmatrix} W\left[e^{c[1]}\right] \\ t[1] \end{pmatrix}\right\}$$

Reversing a Strand

```
{R[1, 2], R[1, 2] // dS[1] , R[1, 2] // dS[2], RInv[1, 2]} // βForm
```

$$\left\{\begin{pmatrix} W[1] & h[2] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] \\ t[1] & -\frac{e^{-c[1]}\left(-1+e^{c[1]}\right)}{c[1]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] \\ t[1] & -\frac{e^{-c[1]}\left(-1+e^{c[1]}\right)}{c[1]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] \\ t[1] & -\frac{e^{-c[1]}\left(-1+e^{c[1]}\right)}{c[1]} \end{pmatrix}\right\}$$

```
{(R[1, 2] // dS[2]) ** R[1, 2], (R[1, 2] // dS[1]) ** R[1, 2], RInv[1, 2] ** R[1, 2]}
```

$$\{W[1], W[1], W[1]\}$$

```
{RInv[1, 1], R[1, 1] ** RInv[1, 1]}
```

$$\left\{\frac{\left(-1+e^{-c[1]}\right) h[1]\, t[1]}{c[1]} + W[1], W[1]\right\}$$

R3

```
{t1 = R[1, 2] ** R[1, 3] ** R[2, 3], t2 = R[2, 3] ** R[1, 3] ** R[1, 2], t1 == t2} // βForm
```

$$\left\{\begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{e^{c[2]}\left(-1+e^{c[1]}\right)}{c[1]} \\ t[2] & 0 & \frac{-1+e^{c[2]}}{c[2]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{e^{c[2]}\left(-1+e^{c[1]}\right)}{c[1]} \\ t[2] & 0 & \frac{-1+e^{c[2]}}{c[2]} \end{pmatrix}, \begin{pmatrix} \text{True} \end{pmatrix}\right\}$$

“Easy” R4

```
{t1 = R[2, 3] ** dΔ[2, 2, 3][R[1, 2]],
  t2 = dΔ[2, 2, 3][R[1, 2]] ** R[2, 3], t1 == t2} // βForm
```

$$\left\{\begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{e^{c[2]}\left(-1+e^{c[1]}\right)}{c[1]} \\ t[2] & 0 & \frac{-1+e^{c[2]}}{c[2]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{e^{c[2]}\left(-1+e^{c[1]}\right)}{c[1]} \\ t[2] & 0 & \frac{-1+e^{c[2]}}{c[2]} \end{pmatrix}, \begin{pmatrix} \text{True} \end{pmatrix}\right\}$$

“Hard” R4

```
{R[1, 2] ** dd[1][R[1, 2]], dd[1][R[1, 2]] ** R[1, 2]} // βForm
```

$$\left\{\begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{e^{-c[1]}\left(-1+e^{c[1]+c[2]}\right)\left(e^{c[1]}c[1]-c[2]+e^{c[1]}c[2]\right)}{c[1]\,(c[1]+c[2])} \\ t[2] & 0 & \frac{e^{-c[1]}\left(-1+e^{c[1]+c[2]}\right)}{c[1]+c[2]} \end{pmatrix}, \begin{pmatrix} W[1] & h[2] & h[3] \\ t[1] & \frac{-1+e^{c[1]}}{c[1]} & \frac{-1+e^{c[1]+c[2]}}{c[1]+c[2]} \\ t[2] & 0 & \frac{-1+e^{c[1]+c[2]}}{c[1]+c[2]} \end{pmatrix}\right\}$$

```
Clear[α, β, γ, δ];
(V = W[ω[c[1], c[2]]] + α[c[1], c[2]] ar[1, 1] + β[c[1], c[2]] ar[1, 2] +
    γ[c[1], c[2]] ar[2, 1] + δ[c[1], c[2]] ar[2, 2]) // βForm
```

$$\begin{pmatrix} W[\omega[c[1], c[2]]] & h[1] & h[2] \\ t[1] & \alpha[c[1], c[2]] & \beta[c[1], c[2]] \\ t[2] & \gamma[c[1], c[2]] & \delta[c[1], c[2]] \end{pmatrix}$$

```
{t1 = V ** dd[1][R[1, 2]], t2 = R[1, 3] ** R[2, 3] ** V} /. c[s_] :> c_s // βForm
```

$$\left\{\begin{pmatrix} W[\omega[c_1, c_2]] & h[1] & h[2] & h[3] \\ t[1] & \alpha[c_1, c_2] & \beta[c_1, c_2] & \frac{(-1+e^{c_1+c_2})\left(1+c_1\,\alpha[c_1,c_2]+c_1\,\beta[c_1,c_2]+c_2\,\beta[c_1,c_2]+c_1^2\,\alpha[c_1,c_2]\,\beta[c_1,c_2]+c_1\,c_2\,\alpha[c_1, \ldots\right.}{(c_1+c_2)\,(1+c_1\,\alpha[c_1,c_2]+c_2\,\gamma[c_1,c_2])\,(\ldots} \\ t[2] & \gamma[c_1, c_2] & \delta[c_1, c_2] & \frac{(-1+e^{c_1+c_2})\left(1+c_1\,\alpha[c_1,c_2]+c_1\,\gamma[c_1,c_2]+c_2\,\gamma[c_1,c_2]+c_1^2\,\beta[c_1,c_2]\,\gamma[c_1,c_2]+c_2\,\delta[c_1,c_2 \ldots\right.}{(c_1+c_2)\,(1+c_1\,\alpha[c_1,c_2]+c_2\,\gamma[c_1,c_2])\,(\ldots} \end{pmatrix}\right.$$

```
eqns1 = βEquations[t1 == t2]
```

$$\Big\{\text{True}, \text{True}, \text{True}, \text{True},$$

$$\Big(\big(-1+e^{c[1]+c[2]}\big)\big(1+c[1]\,\alpha[c[1],c[2]]+c[1]\,\beta[c[1],c[2]]+c[2]\,\beta[c[1],c[2]]+c[1]^2\,\alpha[c[1],c[2]]\,\beta[c[1],c[2]]+c[1]\,c[2]\,\alpha[c[1],c[2]]\,\beta[c[1],c[2]]+c[2]^2\,\beta[c[1],c[2]]\,\gamma[c[1],c[2]]+c[2]\,\delta[c[1],c[2]]+c[1]\,c[2]\,\alpha[c[1],c[2]]\,\delta[c[1],c[2]]\big)\Big)\Big/ \Big((c[1]+c[2])\,(1+c[1]\,\alpha[c[1],c[2]]+c[2]\,\gamma[c[1],c[2]])\,(1+c[1]\,\beta[c[1],c[2]]+c[2]\,\delta[c[1],c[2]])\Big) == \frac{-1+e^{c[1]}}{c[1]},$$

$$\Big(\big(-1+e^{c[1]+c[2]}\big)\big(1+c[1]\,\alpha[c[1],c[2]]+c[1]\,\gamma[c[1],c[2]]+c[2]\,\gamma[c[1],c[2]]+c[1]^2\,\beta[c[1],c[2]]\,\gamma[c[1],c[2]]+c[2]\,\delta[c[1],c[2]]+c[1]\,c[2]\,\alpha[c[1],c[2]]\,\delta[c[1],c[2]]+c[1]\,c[2]\,\gamma[c[1],c[2]]\,\delta[c[1],c[2]]+c[2]^2\,\gamma[c[1],c[2]]\,\delta[c[1],c[2]]\big)\Big)\Big/ \Big((c[1]+c[2])\,(1+c[1]\,\alpha[c[1],c[2]]+c[2]\,\gamma[c[1],c[2]])\,(1+c[1]\,\beta[c[1],c[2]]+c[2]\,\delta[c[1],c[2]])\Big) == \frac{e^{c[1]}\big(-1+e^{c[2]}\big)}{c[2]}, \text{True}\Big\}$$

```
eqns1 /. (ϵ : (α | β | γ | δ))[c[1], c[2]] :> ϵ /. c[s_] :> c_s
```

$$\Big\{\text{True}, \text{True}, \text{True}, \text{True},$$

$$\Big((-1+e^{c_1+c_2})\big(1+\alpha\,c_1+\beta\,c_1+\alpha\,\beta\,c_1^2+\beta\,c_2+\delta\,c_2+\alpha\,\beta\,c_1\,c_2+\alpha\,\delta\,c_1\,c_2+\beta\,\gamma\,c_2^2\big)\Big)\Big/ \big((c_1+c_2)\,(1+\alpha\,c_1+\gamma\,c_2)\,(1+\beta\,c_1+\delta\,c_2)\big) == \frac{-1+e^{c_1}}{c_1},$$

$$\Big((-1+e^{c_1+c_2})\big(1+\alpha\,c_1+\gamma\,c_1+\beta\,\gamma\,c_1^2+\gamma\,c_2+\delta\,c_2+\alpha\,\delta\,c_1\,c_2+\gamma\,\delta\,c_1\,c_2+\gamma\,\delta\,c_2^2\big)\Big)\Big/ \big((c_1+c_2)\,(1+\alpha\,c_1+\gamma\,c_2)\,(1+\beta\,c_1+\delta\,c_2)\big) == \frac{e^{c_1}\,(-1+e^{c_2})}{c_2}, \text{True}\Big\}$$

```
(sol = Solve[eqns1 /. (ϵ : (α | β | γ | δ))[c[1], c[2]] :> ϵ, {α, β, γ, δ}]) /. c[s_] :> cs
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\left\{\left\{\gamma \to \left(-\frac{e^{c_1}}{c_2^2}+\frac{e^{c_1+c_2}}{c_2^2}-\frac{e^{c_1}\,\alpha\, c_1}{c_2^2}+\frac{e^{c_1+c_2}\,\alpha\, c_1}{c_2^2}-\frac{e^{c_1}\,\beta\, c_1}{c_2^2}+\frac{e^{c_1+c_2}\,\beta\, c_1}{c_2^2}-\frac{e^{c_1}\,\alpha\,\beta\, c_1^2}{c_2^2}+\frac{e^{c_1+c_2}\,\alpha\,\beta\, c_1^2}{c_2^2}+\frac{\alpha}{c_2}-\frac{e^{c_1+c_2}\,\alpha}{c_2}-\frac{e^{c_1}\,\delta}{c_2}+\frac{e^{c_1+c_2}\,\delta}{c_2}-\frac{e^{c_1}\,\alpha\,\delta\, c_1}{c_2}+\frac{e^{c_1+c_2}\,\alpha\,\delta\, c_1}{c_2}-\frac{\alpha}{c_1+c_2}+\frac{e^{c_1+c_2}\,\alpha}{c_1+c_2}+\frac{\delta}{c_1+c_2}-\frac{e^{c_1+c_2}\,\delta}{c_1+c_2}+\frac{\alpha\,\delta\, c_1}{c_1+c_2}-\frac{e^{c_1+c_2}\,\alpha\,\delta\, c_1}{c_1+c_2}+\frac{1}{c_2\,(c_1+c_2)}-\frac{e^{c_1+c_2}}{c_2\,(c_1+c_2)}\right)\Big/\left(-\delta+e^{c_1}\,\delta-\frac{1}{c_2}+\frac{e^{c_1}}{c_2}-\frac{\beta\, c_1}{c_2}+\frac{e^{c_1}\,\beta\, c_1}{c_2}+\frac{\beta\, c_1}{c_1+c_2}-\frac{e^{c_1+c_2}\,\beta\, c_1}{c_1+c_2}\right)\right\}\right\}$$

```
(γsol = (sol /. {α | β | δ → 0})[[1, 1, 2]] // FullSimplify) /. c[s_] :> cs
```

$$\frac{e^{c_1}\,\left(\left(-1+e^{c_2}\right)\,c_1-c_2\right)+c_2}{\left(-1+e^{c_1}\right)\,c_2\,\left(c_1+c_2\right)}$$

```
FullSimplify[eqns1 /. {_α | _β | _δ → 0, _γ → γsol}]
```

{True, True, True, True, True, True, True}

- **Θ**

```
(Θ = (R[1, 1, 1 / 2] // dΔ[1, 1, 2]) ** R[1, 1, -1 / 2] ** R[2, 2, -1 / 2]) /. c[s_] :> cs //
 βForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \dfrac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}}\,c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1+c_2-e^{\frac{c_1}{2}}\,c_2\right)}{c_1\,(c_1+c_2)} & \dfrac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} \\ t[2] & \dfrac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \dfrac{e^{-\frac{c_2}{2}}\left(c_1-e^{\frac{c_2}{2}}\,c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_2-e^{\frac{c_2}{2}}\,c_2\right)}{c_2\,(c_1+c_2)} \end{pmatrix}$$

```
(Θ // DeWheel // FullSimplify) /. Log[Exp[x_]] :> x /. c[s_] :> cs // βForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \dfrac{e^{-\frac{c_1}{2}}\,c_2\left(-e^{\frac{c_1}{2}}\,c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_1+c_2-e^{\frac{c_1}{2}}\,c_2\right)}{2\left(-1+e^{\frac{c_2}{2}}\right)c_1\,(c_1+c_2)} & \dfrac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)c_1}{2\left(-1+e^{\frac{c_1}{2}}\right)(c_1+c_2)} \\ t[2] & \dfrac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)c_2}{2\left(-1+e^{\frac{c_2}{2}}\right)(c_1+c_2)} & -\dfrac{e^{-\frac{c_2}{2}}\,c_1\left(-c_1+e^{\frac{c_2}{2}}\,c_1-e^{\frac{c_1}{2}+\frac{c_2}{2}}\,c_2+e^{\frac{c_2}{2}}\,c_2\right)}{2\left(-1+e^{\frac{c_1}{2}}\right)c_2\,(c_1+c_2)} \end{pmatrix}$$

```
{(Θ // dP[1 → 2, 2 → 1]) == Θ, (Θ // dS[1] // dS[2]) == Θ}
```

{True, True}

```
{
    t1 = Θ ** (R[1, 2] // dd[1]),
    t2 = (R[1, 2] // dd[1]) ** Θ,
    t1 == t2 // Simplify
   } /. c[s_] :> c_s // βForm // ColumnForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \frac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1+c_2-e^{\frac{c_1}{2}} c_2\right)}{c_1 (c_1+c_2)} & \frac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)\left(1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} \\ t[2] & \frac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{e^{-\frac{c_2}{2}}\left(c_1-e^{\frac{c_2}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2-e^{\frac{c_2}{2}} c_2\right)}{c_2 (c_1+c_2)} & \frac{\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)\left(1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \frac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1+c_2-e^{\frac{c_1}{2}} c_2\right)}{c_1 (c_1+c_2)} & \frac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \\ t[2] & \frac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{e^{-\frac{c_2}{2}}\left(c_1-e^{\frac{c_2}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2-e^{\frac{c_2}{2}} c_2\right)}{c_2 (c_1+c_2)} & \frac{-1+e^{c_1+c_2}}{c_1+c_2} \end{pmatrix}$$

(True)

```
{
    t1 = Θ ** (R[2, 1] // dd[1]),
    t2 = (R[2, 1] // dd[1]) ** Θ,
    t1 == t2 // Simplify
   } /. c[s_] :> c_s  // βForm // ColumnForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \frac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1+c_2-e^{\frac{c_1}{2}} c_2\right)}{c_1 (c_1+c_2)} & \frac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} \\ t[2] & \frac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{e^{-\frac{c_2}{2}}\left(c_1-e^{\frac{c_2}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2-e^{\frac{c_2}{2}} c_2\right)}{c_2 (c_1+c_2)} \\ t[3] & \frac{e^{\frac{c_2}{2}} (-1+e^{c_3})}{c_3} & \frac{e^{\frac{c_1}{2}} (-1+e^{c_3})}{c_3} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \frac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1+c_2-e^{\frac{c_1}{2}} c_2\right)}{c_1 (c_1+c_2)} & \frac{e^{-\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} \\ t[2] & \frac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}\right)}{c_1+c_2} & \frac{e^{-\frac{c_2}{2}}\left(c_1-e^{\frac{c_2}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2-e^{\frac{c_2}{2}} c_2\right)}{c_2 (c_1+c_2)} \\ t[3] & \frac{e^{\frac{c_2}{2}} (-1+e^{c_3})}{c_3} & \frac{e^{\frac{c_1}{2}} (-1+e^{c_3})}{c_3} \end{pmatrix}$$

(True)

```
{Θ // dη[1], Θ // dη[2]} // βForm
```

{ (W[1]), (W[1]) }

```
Limit[Θ /. {W[_] → 0, c[s_] :> ϵ c[s]}, ϵ → 0]
```

$$\frac{1}{2} (h[2]\, t[1] + h[1]\, t[2])$$

■ The Twist Equation

```
{
  (V // dP[2, 1]) ** Θ,
  R[1, 2] ** V
 } /. c[s_] :> c_s // βForm
```

$$\left\{\begin{pmatrix} W[\omega[c_2, c_1]] & \\ t[1] & \frac{e^{-\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} c_1+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1+c_2-e^{\frac{c_1}{2}} c_2-e^{\frac{c_1}{2}} c_1 c_2\,\alpha[c_2,c_1]+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1 c_2\,\alpha[c_2,c_1]+c_2^2\,\alpha[c_2,c_1]-e^{\frac{c_1}{2}} c_2^2\,\alpha[c_2,c_1]-e^{\frac{c_1}{2}} c_1^2\,\gamma[c_2, \ldots\right.}{\ldots} \\ t[2] & \frac{e^{-\frac{c_1}{2}}\left(-1+e^{\frac{c_1}{2}+\frac{c_2}{2}}-c_2\,\alpha[c_2,c_1]+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_2\,\alpha[c_2,c_1]+e^{\frac{c_1}{2}+\frac{c_2}{2}} c_1\,\beta[c_2,c_1]+\ldots\right.}{\ldots}\end{pmatrix}\right.$$

$$\left.\begin{pmatrix} W[\omega[c_1, c_2]] & h[1] & h[2] \\ t[1] & \frac{e^{-c_1}\left(e^{c_1} c_1\,\alpha[c_1,c_2]-c_2\,\gamma[c_1,c_2]+e^{c_1} c_2\,\gamma[c_1,c_2]\right)}{c_1} & \frac{-1+e^{c_1}+e^{c_1} c_1\,\beta[c_1,c_2]-c_2\,\delta[c_1,c_2]+e^{c_1} c_2\,\delta[c_1,c_2]}{c_1} \\ t[2] & e^{-c_1}\,\gamma[c_1, c_2] & \delta[c_1, c_2]\end{pmatrix}\right\}$$

```
(eqns2 = βEquations[(V // dP[2, 1]) ** Θ == R[1, 2] ** V] // FullSimplify) /. c[s_] :> c_s
```

$$\Big\{\frac{1}{c_1 (c_1+c_2)\,(1+c_2\,\alpha[c_2,c_1]+c_1\,\gamma[c_2,c_1])}\, e^{-\frac{c_1}{2}}\left(e^{\frac{c_1}{2}}\left(\left(-1+e^{\frac{c_2}{2}}\right)c_1-c_2\right)+c_2+c_1\left(e^{\frac{1}{2}(c_1+c_2)}c_1+c_2\right)\delta[c_2,c_1]+\right.$$
$$c_1\,\gamma[c_2,c_1]\left(\left(-1+e^{\frac{1}{2}(c_1+c_2)}\right)c_2^2\,\beta[c_2,c_1]+e^{\frac{c_1}{2}}(c_1+c_2)\left(-1+e^{\frac{c_2}{2}}(1+c_1\,\delta[c_2,c_1])\right)\right)+$$
$$\left.c_2\,\alpha[c_2,c_1]\left(c_2(1+c_1\,\delta[c_2,c_1])+e^{\frac{c_1}{2}}\left(-c_1-c_2+e^{\frac{c_2}{2}}c_1(1+c_1\,\delta[c_2,c_1])\right)\right)\right) ==$$
$$\frac{c_1\,\alpha[c_1,c_2]+e^{-c_1}(-1+e^{c_1})\,c_2\,\gamma[c_1,c_2]}{c_1},$$

$$\left(e^{-\frac{c_1}{2}}\left(\beta[c_2,c_1]\left(c_1 c_2\,\gamma[c_2,c_1]+e^{\frac{1}{2}(c_1+c_2)}\left(c_1+c_2+c_1^2\,\gamma[c_2,c_1]\right)\right)+\right.\right.$$
$$\left(-1+e^{\frac{1}{2}(c_1+c_2)}\right)(1+c_1\,\delta[c_2,c_1])+$$
$$\left.\left.c_2\,\alpha[c_2,c_1]\left(-1-c_1\,\delta[c_2,c_1]+e^{\frac{1}{2}(c_1+c_2)}\left(1+(c_1+c_2)\,\beta[c_2,c_1]+c_1\,\delta[c_2,c_1]\right)\right)\right)\right)\Big/$$
$$((c_1+c_2)\,(1+c_2\,\alpha[c_2,c_1]+c_1\,\gamma[c_2,c_1])) == e^{-c_1}\,\gamma[c_1,c_2],$$

$$\left(e^{-\frac{c_2}{2}}\left(\left(-1+e^{\frac{1}{2}(c_1+c_2)}\right)(1+c_1\,\delta[c_2,c_1])+\left(-1+e^{\frac{1}{2}(c_1+c_2)}\right)c_2\,\alpha[c_2,c_1]\,(1+c_1\,\delta[c_2,c_1])+\right.\right.$$
$$\left.\left.\gamma[c_2,c_1]\left(c_1 c_2\,\beta[c_2,c_1]+e^{\frac{1}{2}(c_1+c_2)}\left(c_2^2\,\beta[c_2,c_1]+(c_1+c_2)\,(1+c_1\,\delta[c_2,c_1])\right)\right)\right)\right)\Big/$$
$$((c_1+c_2)\,(1+c_2\,\beta[c_2,c_1]+c_1\,\delta[c_2,c_1])) == \frac{e^{c_1} c_1\,\beta[c_1,c_2]+(-1+e^{c_1})\,(1+c_2\,\delta[c_1,c_2])}{c_1},$$

$$\frac{1}{c_2 (c_1+c_2)\,(1+c_2\,\beta[c_2,c_1]+c_1\,\delta[c_2,c_1])}\, e^{-\frac{c_2}{2}}\left(c_2\,\beta[c_2,c_1]\left(e^{\frac{c_2}{2}}\left(-1+e^{\frac{c_1}{2}}\right)(c_1+c_2)+\left(-1+e^{\frac{1}{2}(c_1+c_2)}\right)c_1^2\,\gamma[c_2,c_1]\right)+\right.$$
$$\left(c_1-e^{\frac{c_2}{2}}\left(c_1+c_2-e^{\frac{c_1}{2}}c_2\right)\right)(1+c_1\,\delta[c_2,c_1])+$$
$$\left.c_2\,\alpha[c_2,c_1]\left(c_1(1+c_1\,\delta[c_2,c_1])+e^{\frac{1}{2}(c_1+c_2)}c_2\left(1+(c_1+c_2)\,\beta[c_2,c_1]+c_1\,\delta[c_2,c_1]\right)\right)\right) ==$$
$$\delta[c_1,c_2],\ \omega[c_1,c_2] == \omega[c_2,c_1]\Big\}$$

■ Unitarity

```
(V ** (V // dS[1] // dS[2])) // Short
```

```
W[(1 - c[1] α[-c[1], -c[2]] - c[1] c[2] β[-c[1], -c[2]] γ[-c[1], -c[2]] -
      c[2] δ[-c[1], -c[2]] + c[1] c[2] α[-c[1], -c[2]] δ[-c[1], -c[2]])
    ω[-c[1], -c[2]] ω[c[1], c[2]]] + <<1>> + h[1] (- <<1>> - <<1>>)
```

```
(V ** (V // dA[1] // dA[2])) /. c[s_] :> c_s // βForm
```

$\left(W\left[(1 + c_1 \alpha[c_1, c_2] - c_1 c_2 \beta[c_1, c_2] \gamma[c_1, c_2] + c_2 \delta[c_1, c_2] + c_1 c_2 \alpha[c_1, c_2] \delta[c_1, c_2]) \omega[c_1, c_2]^2 \right]\right.$

```
(eqns3 = βEquations[V ** (V // dA[1] // dA[2]) == W[1]]) /. c[s_] :> c_s
```

$\{(1 + c_1 \alpha[c_1, c_2] - c_1 c_2 \beta[c_1, c_2] \gamma[c_1, c_2] + c_2 \delta[c_1, c_2] + c_1 c_2 \alpha[c_1, c_2] \delta[c_1, c_2]) \omega[c_1, c_2]^2 == 1\}$

■ Non-degeneracy

```
(eqns4 = Join[
    βEquations[(V // dη[1]) == W[1]],
    βEquations[(V // dη[2]) == W[1]]
   ]) /. c[s_] :> c_s
```

$\{\delta[0, c_2] == 0, \omega[0, c_2] == 1, \alpha[c_1, 0] == 0, \omega[c_1, 0] == 1\}$

■ Cap equation

```
{Cap = W[κ[c[1]]],
  Cap // dd[1],
  V ** (Cap // dd[1]),
  (V ** (Cap // dd[1])) // hη[1] // hη[2],
  Cap + (Cap // dP[2])
 } /. c[s_] :> c_s // βForm
```

$\left\{ (W[\kappa[c_1]]), (W[\kappa[c_1 + c_2]]), \begin{pmatrix} W[\kappa[c_1 + c_2] \omega[c_1, c_2]] & h[1] & h[2] \\ t[1] & \alpha[c_1, c_2] & \beta[c_1, c_2] \\ t[2] & \gamma[c_1, c_2] & \delta[c_1, c_2] \end{pmatrix}, (W[\kappa[c_1 + c_2] \omega[c_1, c_2]]), (W[\kappa[c_1] \kappa[c_2]]) \right\}$

```
(eqns5 = βEquations[
    ((V ** (Cap // dd[1])) // hη[1] // hη[2]) == (Cap + (Cap // dP[2]))]) /. c[s_] :> c_s
```

$\{\kappa[c_1 + c_2] \omega[c_1, c_2] == \kappa[c_1] \kappa[c_2]\}$

■ 120 Degrees Rotational Symmetry

```
(eqns6 = βEquations[V == Rot120[V]]) /. c[s_] :> c_s
```

$\{\alpha[c_1, c_2] == -((1 + c_2 \alpha[c_2, -c_1 - c_2] - c_1 \gamma[c_2, -c_1 - c_2] - c_2 \gamma[c_2, -c_1 - c_2]) \delta[c_2, -c_1 - c_2]) /$
$(-1 - c_2 \alpha[c_2, -c_1 - c_2] + c_1 \gamma[c_2, -c_1 - c_2] - c_2^2 \beta[c_2, -c_1 - c_2] \gamma[c_2, -c_1 - c_2] +$
$c_1 \delta[c_2, -c_1 - c_2] + c_2 \delta[c_2, -c_1 - c_2] + c_1 c_2 \alpha[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] +$
$c_2^2 \alpha[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] - c_1^2 \gamma[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] -$
$c_1 c_2 \gamma[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2]), \gamma[c_1, c_2] ==$
$-(\beta[c_2, -c_1 - c_2] + c_2 \alpha[c_2, -c_1 - c_2] \beta[c_2, -c_1 - c_2] - c_1 \beta[c_2, -c_1 - c_2] \gamma[c_2, -c_1 - c_2] +$
$c_2^2 \beta[c_2, -c_1 - c_2]^2 \gamma[c_2, -c_1 - c_2] - \delta[c_2, -c_1 - c_2] - c_2 \alpha[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] -$
$c_2 \beta[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] - c_2^2 \alpha[c_2, -c_1 - c_2] \beta[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] -$
$c_1 c_2 \beta[c_2, -c_1 - c_2] \gamma[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] -$
$c_2^2 \beta[c_2, -c_1 - c_2] \gamma[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2] + c_1 \delta[c_2, -c_1 - c_2]^2 + c_2 \delta[c_2, -c_1 - c_2]^2 +$
$c_1 c_2 \alpha[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2]^2 + c_2^2 \alpha[c_2, -c_1 - c_2] \delta[c_2, -c_1 - c_2]^2) /$

$$
\begin{aligned}
&((1 + c_2\,\beta[c_2, -c_1 - c_2] - c_1\,\delta[c_2, -c_1 - c_2] - c_2\,\delta[c_2, -c_1 - c_2]) \\
&\quad (1 + c_2\,\alpha[c_2, -c_1 - c_2] - c_1\,\gamma[c_2, -c_1 - c_2] + \\
&\qquad c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] - c_1\,\delta[c_2, -c_1 - c_2] - c_2\,\delta[c_2, -c_1 - c_2] - \\
&\qquad c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - c_2^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\qquad c_1^2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + c_1\,c_2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2])), \\
&\beta[c_1, c_2] == -((1 + c_2\,\alpha[c_2, -c_1 - c_2] - c_1\,\gamma[c_2, -c_1 - c_2] - c_2\,\gamma[c_2, -c_1 - c_2])\,(-\gamma[c_2, -c_1 - c_2] - \\
&\qquad c_2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + \delta[c_2, -c_1 - c_2] + c_2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]))\,/ \\
&\quad (-1 - c_2\,\alpha[c_2, -c_1 - c_2] + c_1\,\gamma[c_2, -c_1 - c_2] - c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + \\
&\qquad c_1\,\delta[c_2, -c_1 - c_2] + c_2\,\delta[c_2, -c_1 - c_2] + \\
&\qquad c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + c_2^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - \\
&\qquad c_1^2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - c_1\,c_2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]), \\
&\delta[c_1, c_2] == (\alpha[c_2, -c_1 - c_2] + c_2\,\alpha[c_2, -c_1 - c_2]^2 - \beta[c_2, -c_1 - c_2] - \\
&\quad c_2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2] - \gamma[c_2, -c_1 - c_2] - c_1\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] - \\
&\quad c_2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + 3\,c_1\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + \\
&\quad 2\,c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + \\
&\quad c_2^2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + c_1\,c_2\,\beta[c_2, -c_1 - c_2]^2\,\gamma[c_2, -c_1 - c_2] - \\
&\quad c_2^2\,\beta[c_2, -c_1 - c_2]^2\,\gamma[c_2, -c_1 - c_2] + c_1\,c_2^2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2]^2\,\gamma[c_2, -c_1 - c_2] - \\
&\quad 2\,c_1^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2 - 3\,c_1\,c_2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2 - \\
&\quad c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2 - c_1^2\,c_2\,\beta[c_2, -c_1 - c_2]^2\,\gamma[c_2, -c_1 - c_2]^2 - \\
&\quad c_1\,c_2^2\,\beta[c_2, -c_1 - c_2]^2\,\gamma[c_2, -c_1 - c_2]^2 + \delta[c_2, -c_1 - c_2] - 2\,c_1\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - \\
&\quad 2\,c_1\,c_2\,\alpha[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2] - c_2^2\,\alpha[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_2\,\beta[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_2^2\,\alpha[c_2, -c_1 - c_2]\,\beta[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - \\
&\quad c_1\,c_2^2\,\alpha[c_2, -c_1 - c_2]^2\,\beta[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + c_1\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + 2\,c_1^2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad 3\,c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_2^2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - \\
&\quad c_1^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_1^3\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2] + \\
&\quad 2\,c_1^2\,c_2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2] + \\
&\quad c_1\,c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2] - c_1\,\delta[c_2, -c_1 - c_2]^2 - \\
&\quad c_2\,\delta[c_2, -c_1 - c_2]^2 + c_1^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]^2 - \\
&\quad c_2^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]^2 + c_1^2\,c_2\,\alpha[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2]^2 + \\
&\quad c_1\,c_2^2\,\alpha[c_2, -c_1 - c_2]^2\,\delta[c_2, -c_1 - c_2]^2 - c_1^3\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]^2 - \\
&\quad 2\,c_1^2\,c_2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]^2 - \\
&\quad c_1\,c_2^2\,\alpha[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2]^2)\,/ \\
&\quad ((1 + c_2\,\beta[c_2, -c_1 - c_2] - c_1\,\delta[c_2, -c_1 - c_2] - c_2\,\delta[c_2, -c_1 - c_2]) \\
&\qquad (1 + c_2\,\alpha[c_2, -c_1 - c_2] - c_1\,\gamma[c_2, -c_1 - c_2] + \\
&\qquad c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] - c_1\,\delta[c_2, -c_1 - c_2] - c_2\,\delta[c_2, -c_1 - c_2] - \\
&\qquad c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - c_2^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\qquad c_1^2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + c_1\,c_2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2])), \\
&\omega[c_1, c_2] == -((-1 - c_2\,\alpha[c_2, -c_1 - c_2] + c_1\,\gamma[c_2, -c_1 - c_2] - c_2^2\,\beta[c_2, -c_1 - c_2]\,\gamma[c_2, -c_1 - c_2] + \\
&\qquad c_1\,\delta[c_2, -c_1 - c_2] + c_2\,\delta[c_2, -c_1 - c_2] + c_1\,c_2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] + \\
&\qquad c_2^2\,\alpha[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - c_1^2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2] - \\
&\qquad c_1\,c_2\,\gamma[c_2, -c_1 - c_2]\,\delta[c_2, -c_1 - c_2])\,\omega[c_2, -c_1 - c_2])\,/ \\
&\quad (1 + c_2\,\alpha[c_2, -c_1 - c_2] - c_1\,\gamma[c_2, -c_1 - c_2] - c_2\,\gamma[c_2, -c_1 - c_2])\}
\end{aligned}
$$

■ Solving the Equations

```
Series[f[τ x, τ y], {τ, 0, 3}] /. f^(i__)[___] :> f_FromDigits[{i}]
```

$$f[0,0]+(y f_1+x f_{10})\,\tau+\frac{1}{2}\left(y^2 f_2+2\,x\,y\,f_{11}+x^2 f_{20}\right)\tau^2+\frac{1}{6}\left(y^3 f_3+3\,x\,y^2 f_{12}+3\,x^2\,y\,f_{21}+x^3 f_{30}\right)\tau^3+O[\tau]^4$$

```
eqns = Join[eqns1, eqns2, eqns3, eqns4, eqns5, eqns6];
```

```
n = 0;
Simplify[Normal[
   Series[
     eqns /. c[s_] :→ τ c[s],
     {τ, 0, n}
    ] /. {(ϵ : (α | β | γ | δ | ω | κ))[___] :→ ϵ0,
     (ϵ : (α | β | γ | δ | ω | κ))^(k__)[___] :→ ϵ_FromDigits[{k}]}
  ]] /.
 c[
   s_] :→
  c_s
```

$$\left\{\text{True},\text{True},\text{True},\text{True},\text{True},\text{True},\text{True},\alpha_0=\delta_0,\frac{1}{2}+\beta_0=\gamma_0,1+2\beta_0=2\gamma_0,\alpha_0=\delta_0,\text{True},\omega_0^2=1,\delta_0=0,\omega_0=1,\alpha_0=0,\omega_0=1,\kappa_0\,\omega_0=\kappa_0^2,\alpha_0=\delta_0,\beta_0+\gamma_0=\delta_0,\beta_0+\gamma_0=\delta_0,\alpha_0=\beta_0+\gamma_0,\text{True}\right\}$$

```
n = 0;
sol = SolveAlways[
  Expand[Normal[
    Series[
      eqns /. c[s_] :→ τ c[s],
      {τ, 0, n}
     ] /. {(ϵ : (α | β | γ | δ | ω | κ))[___] :→ ϵ0,
      (ϵ : (α | β | γ | δ | ω | κ))^(k__)[___] :→ ϵ_FromDigits[{k}]}
   ]],
  {c[1], c[2], τ}
 ]
```

$$\left\{\left\{\alpha_0\to 0,\beta_0\to -\frac{1}{4},\gamma_0\to \frac{1}{4},\delta_0\to 0,\kappa_0\to 0,\omega_0\to 1\right\},\left\{\alpha_0\to 0,\beta_0\to -\frac{1}{4},\gamma_0\to \frac{1}{4},\delta_0\to 0,\kappa_0\to 1,\omega_0\to 1\right\}\right\}$$

```
If[Length[sol] == 2,
  {Complement[sol[[1]], sol[[2]]], Complement[sol[[2]], sol[[1]]]}] // ColumnForm
```

$\{\kappa_0\to 0\}$

$\{\kappa_0\to 1\}$

```
indvars = Union[Cases[Last /@ #, ϵ_k_ :→ ϵ_k, Infinity]] & /@ sol
```

{{}, {}}

```
sol0 = ((# → 0) & /@ First[indvars]);
sol0 = Union[sol0, First[sol] /. sol0]
```

$\{\alpha_0 \to 0,\ \beta_0 \to -\frac{1}{4},\ \gamma_0 \to \frac{1}{4},\ \delta_0 \to 0,\ \kappa_0 \to 0,\ \omega_0 \to 1\}$

```
(Normal[
    Series[
      V /. {W[_] → 0, c[s_] :→ τ c[s]},
      {τ, 0, n}
     ] /.
     {(ϵ : (α | β | γ | δ | ω))[___] :→ ϵ0, (ϵ : (α | β | γ | δ | ω))^(k__)[___] :→ ϵFromDigits[{k}]}
   ] /. sol0) /. c[s_] :→ cs // βForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] \\ t[1] & 0 & -\frac{1}{4} \\ t[2] & \frac{1}{4} & 0 \end{pmatrix}$$

```
(Normal[
    Series[
      ω[c[1], c[2]] /. {W[_] → 0, c[s_] :→ τ c[s]},
      {τ, 0, n}
     ] /.
     {(ϵ : (α | β | γ | δ | ω))[___] :→ ϵ0, (ϵ : (α | β | γ | δ | ω))^(k__)[___] :→ ϵFromDigits[{k}]}
   ] /. sol0) /. c[s_] :→ cs // βForm
```

$$\begin{pmatrix} 1 \end{pmatrix}$$