Pensieve Header: The braid group action that arises in β calculus and the n-strand group.

In[155]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-01"];
<< betaCalculus.m;
βSimplify = FullSimplify;
```

In[19]:=
```
ar[1, 2] ** (-1/(1 + c[1])) ar[1, 2]) // βForm
```

Out[19]//MatrixForm=

$$\begin{pmatrix} 0 \end{pmatrix}$$

In[164]:=
```
{β1 = W[1] + Sum[α_{10 i+j} ar[i, j], {i, 3}, {j, 3}] + ar[5, 6],
  β1 // dm[1, 5, 1],
  β1 // dm[2, 6, 2],
  β1 // dm[1, 5, 1] // dm[2, 6, 2]
 } /. c[i_] :> c_i // βForm // ColumnForm
```

Out[164]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] & h[6] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 \\ t[5] & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] & h[6] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} & \frac{1+c_1\alpha_{11}}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} & \frac{c_1\alpha_{21}}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} & \frac{c_1\alpha_{31}}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \\ t[5] & 0 & 1 + c_1\alpha_{12} + c_2\alpha_{22} + c_3\alpha_{32} & 0 \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} + \frac{(1+c_1\alpha_{11})(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} + \frac{c_1\alpha_{21}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} + \frac{c_1\alpha_{31}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{33} \end{pmatrix}$$

In[158]:=
```
{β1 = W[1] + Sum[α10 i+j ar[i, j], {i, 3}, {j, 3}],
    β1 ** ar[1, 2],
    β1 ** (-1/(1 + c[1])) ar[1, 2]
  } /. c[i_] :> c_i // βForm // ColumnForm
```

Out[158]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12}+\frac{(1+c_1\alpha_{11})(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22}+\frac{c_1\alpha_{21}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32}+\frac{c_1\alpha_{31}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31}} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12}-\frac{(1+c_1\alpha_{11})(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{(1+c_1)(1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31})} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22}-\frac{c_1\alpha_{21}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{(1+c_1)(1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31})} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32}-\frac{c_1\alpha_{31}(1+c_1\alpha_{12}+c_2\alpha_{22}+c_3\alpha_{32})}{(1+c_1)(1+c_1\alpha_{11}+c_2\alpha_{21}+c_3\alpha_{31})} & \alpha_{33} \end{pmatrix}$$

In[23]:=
```
{β1 = W[1] + Sum[α10 i+j ar[i, j], {i, 3}, {j, 3}],
    β1 ** ar[1, 2],
    β1 ** (-1/(1 + c[1])) ar[1, 2]
  } /. c[i_] :> c // βForm // ColumnForm
```

Out[23]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12}+\frac{(1+c\,\alpha_{11})(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31})} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22}+\frac{c\,\alpha_{21}(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31})} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32}+\frac{c\,\alpha_{31}(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31})} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12}-\frac{(1+c\,\alpha_{11})(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{(1+c)(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22}-\frac{c\,\alpha_{21}(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{(1+c)(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32}-\frac{c\,\alpha_{31}(1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32}))}{(1+c)(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))} & \alpha_{33} \end{pmatrix}$$

In[165]:=
```
{β1 = W[1] + Sum[α10 i+j ar[i, j], {i, 3}, {j, 3}],
   ar[1, 2] ** β1,
   (-1/(1 + c[1])) ar[1, 2]) ** β1
  } /. c[i_] :> c_i // βForm // ColumnForm
```

Out[165]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} + \frac{c_2\,\alpha_{21}}{1+c_1} & 1 + (1+c_1)\,\alpha_{12} + c_2\,\alpha_{22} & \alpha_{13} + \frac{c_2\,\alpha_{23}}{1+c_1} \\ t[2] & \frac{\alpha_{21}}{1+c_1} & \alpha_{22} & \frac{\alpha_{23}}{1+c_1} \\ t[3] & \alpha_{31} & (1+c_1)\,\alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} - c_2\,\alpha_{21} & \frac{-1+\alpha_{12}-c_2\,\alpha_{22}}{1+c_1} & \alpha_{13} - c_2\,\alpha_{23} \\ t[2] & (1+c_1)\,\alpha_{21} & \alpha_{22} & (1+c_1)\,\alpha_{23} \\ t[3] & \alpha_{31} & \frac{\alpha_{32}}{1+c_1} & \alpha_{33} \end{pmatrix}$$

In[166]:=
```
{β1 = W[1] + Sum[α10 i+j ar[i, j], {i, 3}, {j, 3}],
   ar[1, 2] ** β1,
   (-1/(1 + c[1])) ar[1, 2]) ** β1
  } /. c[i_] :> c // βForm // ColumnForm
```

Out[166]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} + \frac{c\,\alpha_{21}}{1+c} & 1 + (1+c)\,\alpha_{12} + c\,\alpha_{22} & \alpha_{13} + \frac{c\,\alpha_{23}}{1+c} \\ t[2] & \frac{\alpha_{21}}{1+c} & \alpha_{22} & \frac{\alpha_{23}}{1+c} \\ t[3] & \alpha_{31} & (1+c)\,\alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} - c\,\alpha_{21} & \frac{-1+\alpha_{12}-c\,\alpha_{22}}{1+c} & \alpha_{13} - c\,\alpha_{23} \\ t[2] & (1+c)\,\alpha_{21} & \alpha_{22} & (1+c)\,\alpha_{23} \\ t[3] & \alpha_{31} & \frac{\alpha_{32}}{1+c} & \alpha_{33} \end{pmatrix}$$

```
n = 1;
{
    α1 = W[1] + Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = W[1] + Sum[β10 i+j ar[i, j], {i, n}, {j, n}],
    α1 ** β1
   } /. c[i_] :> c_i // βForm // ColumnForm
```

$$\begin{pmatrix} W[1] & h[1] \\ t[1] & \alpha_{11} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] \\ t[1] & \beta_{11} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] \\ t[1] & \beta_{11} + \alpha_{11}\,(1 + c_1\,\beta_{11}) \end{pmatrix}$$

```
n = 2;
{
    α1 = W[1] + Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = W[1] + Sum[β10 i+j ar[i, j], {i, n}, {j, n}],
    α1 ** β1
   } /. c[i_] :> c_i // βForm // ColumnForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \beta_{11} & \beta_{12} \\ t[2] & \beta_{21} & \beta_{22} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \beta_{11} + \alpha_{11}\,(1 + c_1\,\beta_{11}) + \frac{c_2\,\alpha_{12}\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,\beta_{21}}{1+c_1\,\alpha_{12}+c_2\,\alpha_{22}} & \alpha_{12} + \frac{(1+c_1\,\alpha_{11})\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})\,\beta_{12}}{1+c_1\,\alpha_{11}+c_2\,\alpha_{21}} + c_2\,\alpha_{12}\,\beta_{22} \\ t[2] & \alpha_{21}\,(1 + c_1\,\beta_{11}) + \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21})\,(1+c_2\,\alpha_{22})\,\beta_{21}}{1+c_1\,\alpha_{12}+c_2\,\alpha_{22}} & \alpha_{22} + \frac{c_1\,\alpha_{21}\,(1+c_1\,\alpha_{12}+c_2\,\alpha_{22})\,\beta_{12}}{1+c_1\,\alpha_{11}+c_2\,\alpha_{21}} + \beta_{22} + c_2\,\alpha_{22}\,\beta_{22} \end{pmatrix}$$

In[24]:=
```
n = 2;
{
    α1 = W[1] + Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = W[1] + Sum[β10 i+j ar[i, j], {i, n}, {j, n}],
    α1 ** β1
   } /. c[i_] :> c // βForm // ColumnForm
```

Out[25]=

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \alpha_{11} & \alpha_{12} \\ t[2] & \alpha_{21} & \alpha_{22} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \beta_{11} & \beta_{12} \\ t[2] & \beta_{21} & \beta_{22} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] \\ t[1] & \beta_{11} + \alpha_{11}\,(1 + c\,\beta_{11}) + \frac{c\,\alpha_{12}\,(1+c\,(\alpha_{11}+\alpha_{21}))\,\beta_{21}}{1+c\,(\alpha_{12}+\alpha_{22})} & \alpha_{12} + \frac{(1+c\,\alpha_{11})\,(1+c\,(\alpha_{12}+\alpha_{22}))\,\beta_{12}}{1+c\,(\alpha_{11}+\alpha_{21})} + c\,\alpha_{12}\,\beta_{22} \\ t[2] & \alpha_{21}\,(1 + c\,\beta_{11}) + \frac{(1+c\,(\alpha_{11}+\alpha_{21}))\,(1+c\,\alpha_{22})\,\beta_{21}}{1+c\,(\alpha_{12}+\alpha_{22})} & \alpha_{22} + \frac{c\,\alpha_{21}\,(1+c\,(\alpha_{12}+\alpha_{22}))\,\beta_{12}}{1+c\,(\alpha_{11}+\alpha_{21})} + \beta_{22} + c\,\alpha_{22}\,\beta_{22} \end{pmatrix}$$

```
n = 3;
{
    α1 = W[1] + Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = W[1] + Sum[β10 i+j ar[i, j], {i, n}, {j, n}],
    α1 ** β1
   } /. c[i_] :> c_i // βForm // ColumnForm
```

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \beta_{11} & \beta_{12} & \beta_{13} \\ t[2] & \beta_{21} & \beta_{22} & \beta_{23} \\ t[3] & \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] \\ t[1] & \beta_{11} + \alpha_{11}\,(1 + c_1\,\beta_{11}) + \frac{c_2\,\alpha_{12}\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\beta_{21}}{1+c_1\,\alpha_{12}+c_2\,\alpha_{22}+c_3\,\alpha_{32}} + \frac{c_3\,\alpha_{13}\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\beta_{31}}{1+c_1\,\alpha_{13}+c_2\,\alpha_{23}+c_3\,\alpha_{33}} \\ t[2] & \alpha_{21}\,(1 + c_1\,\beta_{11}) + \frac{(1+c_2\,\alpha_{22})\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\beta_{21}}{1+c_1\,\alpha_{12}+c_2\,\alpha_{22}+c_3\,\alpha_{32}} + \frac{c_3\,\alpha_{23}\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\beta_{31}}{1+c_1\,\alpha_{13}+c_2\,\alpha_{23}+c_3\,\alpha_{33}} \\ t[3] & \alpha_{31}\,(1 + c_1\,\beta_{11}) + \frac{c_2\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\alpha_{32}\,\beta_{21}}{1+c_1\,\alpha_{12}+c_2\,\alpha_{22}+c_3\,\alpha_{32}} + \frac{(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\beta_{31}}{1+c_1\,\alpha_{13}+c_2\,\alpha_{23}+c_3\,\alpha_{33}} + \frac{c_3\,(1+c_1\,\alpha_{11}+c_2\,\alpha_{21}+c_3\,\alpha_{31})\,\alpha_{33}\,\beta_{31}}{1+c_1\,\alpha_{13}+c_2\,\alpha_{23}+c_3\,\alpha_{33}} \end{pmatrix}$$

In[26]:=
```
n = 3;
{
    α1 = W[1] + Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = W[1] + Sum[β10 i+j ar[i, j], {i, n}, {j, n}],
    α1 ** β1
   } /. c[i_] :> c // βForm // ColumnForm
```

Out[27]=

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] & h[2] & h[3] \\ t[1] & \beta_{11} & \beta_{12} & \beta_{13} \\ t[2] & \beta_{21} & \beta_{22} & \beta_{23} \\ t[3] & \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}$$

$$\begin{pmatrix} W[1] & h[1] \\ t[1] & \beta_{11} + \alpha_{11}\,(1 + c\,\beta_{11}) + \frac{c\,\alpha_{12}\,(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\beta_{21}}{1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32})} + \frac{c\,\alpha_{13}\,(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\beta_{31}}{1+c\,(\alpha_{13}+\alpha_{23}+\alpha_{33})} \\ t[2] & \alpha_{21}\,(1 + c\,\beta_{11}) + \frac{(1+c\,\alpha_{22})\,(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\beta_{21}}{1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32})} + \frac{c\,\alpha_{23}\,(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\beta_{31}}{1+c\,(\alpha_{13}+\alpha_{23}+\alpha_{33})} \\ t[3] & \alpha_{31}\,(1 + c\,\beta_{11}) + \frac{c\,(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\alpha_{32}\,\beta_{21}}{1+c\,(\alpha_{12}+\alpha_{22}+\alpha_{32})} + \frac{\beta_{31}}{1+c\,(\alpha_{13}+\alpha_{23}+\alpha_{33})} + \frac{c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}+(1+c\,(\alpha_{11}+\alpha_{21}+\alpha_{31}))\,\alpha_{33})\,\beta_{31}}{1+c\,(\alpha_{13}+\alpha_{23}+\alpha_{33})} \end{pmatrix}$$

```
n = 2;
{
    α1 = Sum[h α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = Sum[h β10 i+j ar[i, j], {i, n}, {j, n}],
    Limit[(α1 ** β1 - β1 ** α1) / h^2, h → 0]
  } /. c[i_] :→ ci // βForm // ColumnForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] \\ t[1] & h\,\alpha_{11} & h\,\alpha_{12} \\ t[2] & h\,\alpha_{21} & h\,\alpha_{22} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] \\ t[1] & h\,\beta_{11} & h\,\beta_{12} \\ t[2] & h\,\beta_{21} & h\,\beta_{22} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] \\ t[1] & c_2\,(-\alpha_{21}\,\beta_{12}+\alpha_{12}\,\beta_{21}) & c_2\,(-\alpha_{21}\,\beta_{12}+\alpha_{12}\,\beta_{21}) \\ t[2] & c_1\,(\alpha_{21}\,\beta_{12}-\alpha_{12}\,\beta_{21}) & c_1\,(\alpha_{21}\,\beta_{12}-\alpha_{12}\,\beta_{21}) \end{pmatrix}$$

```
n = 3; βSimplify = Factor;
{
    α1 = Sum[ϵ α10 i+j ar[i, j], {i, n}, {j, n}],
    β1 = Sum[ϵ β10 i+j ar[i, j], {i, n}, {j, n}],
    bracket = βCollect[(α1 ** β1 - β1 ** α1) / ϵ^2] /. ϵ → 0
  } /. c[i_] :→ ci // βForm // ColumnForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & \epsilon\,\alpha_{11} & \epsilon\,\alpha_{12} & \epsilon\,\alpha_{13} \\ t[2] & \epsilon\,\alpha_{21} & \epsilon\,\alpha_{22} & \epsilon\,\alpha_{23} \\ t[3] & \epsilon\,\alpha_{31} & \epsilon\,\alpha_{32} & \epsilon\,\alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & \epsilon\,\beta_{11} & \epsilon\,\beta_{12} & \epsilon\,\beta_{13} \\ t[2] & \epsilon\,\beta_{21} & \epsilon\,\beta_{22} & \epsilon\,\beta_{23} \\ t[3] & \epsilon\,\beta_{31} & \epsilon\,\beta_{32} & \epsilon\,\beta_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & \\ t[1] & -c_2\,\alpha_{21}\,\beta_{12}-c_3\,\alpha_{31}\,\beta_{13}+c_2\,\alpha_{12}\,\beta_{21}+c_3\,\alpha_{13}\,\beta_{31} & -c_2\,\alpha_{2} \\ t[2] & c_1\,\alpha_{21}\,\beta_{12}-c_1\,\alpha_{12}\,\beta_{21}+c_3\,\alpha_{31}\,\beta_{21}-c_3\,\alpha_{32}\,\beta_{21}-c_3\,\alpha_{31}\,\beta_{23}-c_3\,\alpha_{21}\,\beta_{31}+c_3\,\alpha_{23}\,\beta_{31}+c_3\,\alpha_{21}\,\beta_{32} & \\ t[3] & c_1\,\alpha_{31}\,\beta_{13}-c_2\,\alpha_{31}\,\beta_{21}+c_2\,\alpha_{32}\,\beta_{21}+c_2\,\alpha_{31}\,\beta_{23}-c_1\,\alpha_{13}\,\beta_{31}+c_2\,\alpha_{21}\,\beta_{31}-c_2\,\alpha_{23}\,\beta_{31}-c_2\,\alpha_{21}\,\beta_{32} & c_1\,\alpha_{3} \end{pmatrix}$$

```
bracket /. {α13 → 1, β23 → 1, α_ → 0, β_ → 0} // βForm
```

$$\begin{pmatrix} 0 & h[3] \\ t[1] & -c[2] \\ t[2] & c[1] \end{pmatrix}$$

```
bracket /. {α12 → 1, β23 → 1, α_ → 0, β_ → 0} // βForm
```

$$\begin{pmatrix} 0 & h[3] \\ t[1] & c[2] \\ t[2] & -c[1] \end{pmatrix}$$

```
bracket /. {α12 → 1, β13 → 1, α_ → 0, β_ → 0} // βForm
```

$$(\,0\,)$$

```
bracket /. {α12 → 1, β21 → 1, α_ → 0, β_ → 0} // βForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] \\ t[1] & c[2] & c[2] \\ t[2] & -c[1] & -c[1] \end{pmatrix}$$

```
n = 3;
{
    α1 = Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    α1 // dS[1],
    βCollect[(ϵ α1 // dS[1]) / ϵ] /. ϵ → 0
   } /. c[i_] :> ci // βForm // ColumnForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & -\frac{\alpha_{11}}{-1+c_1 \alpha_{11}} & -\frac{\alpha_{12}\,(-1+c_1 \alpha_{11}-c_2 \alpha_{21}-c_3 \alpha_{31})}{-1+c_1 \alpha_{11}} & -\frac{\alpha_{13}\,(-1+c_1 \alpha_{11}-c_2 \alpha_{21}-c_3 \alpha_{31})}{-1+c_1 \alpha_{11}} \\ t[2] & -\frac{\alpha_{21}}{(-1+c_1 \alpha_{11})\,(-1+c_1 \alpha_{11}-c_2 \alpha_{21}-c_3 \alpha_{31})} & \frac{-c_1 \alpha_{12} \alpha_{21}-\alpha_{22}+c_1 \alpha_{11} \alpha_{22}}{-1+c_1 \alpha_{11}} & \frac{-c_1 \alpha_{13} \alpha_{21}-\alpha_{23}+c_1 \alpha_{11} \alpha_{23}}{-1+c_1 \alpha_{11}} \\ t[3] & -\frac{\alpha_{31}}{(-1+c_1 \alpha_{11})\,(-1+c_1 \alpha_{11}-c_2 \alpha_{21}-c_3 \alpha_{31})} & \frac{-c_1 \alpha_{12} \alpha_{31}-\alpha_{32}+c_1 \alpha_{11} \alpha_{32}}{-1+c_1 \alpha_{11}} & \frac{-c_1 \alpha_{13} \alpha_{31}-\alpha_{33}+c_1 \alpha_{11} \alpha_{33}}{-1+c_1 \alpha_{11}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ t[2] & -\alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & -\alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

```
n = 3;
{
    α1 = Sum[α10 i+j ar[i, j], {i, n}, {j, n}],
    α1 // dA[1],
    βCollect[(ϵ α1 // dA[1]) / ϵ] /. ϵ → 0
   } /. c[i_] :> ci // βForm // ColumnForm
```

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & -\frac{\alpha_{11}}{1+c_1 \alpha_{11}} & \frac{\alpha_{12}\,(1+c_1 \alpha_{11}+c_2 \alpha_{21}+c_3 \alpha_{31})}{1+c_1 \alpha_{11}} & \frac{\alpha_{13}\,(1+c_1 \alpha_{11}+c_2 \alpha_{21}+c_3 \alpha_{31})}{1+c_1 \alpha_{11}} \\ t[2] & -\frac{\alpha_{21}}{(1+c_1 \alpha_{11})\,(1+c_1 \alpha_{11}+c_2 \alpha_{21}+c_3 \alpha_{31})} & \frac{-c_1 \alpha_{12} \alpha_{21}+\alpha_{22}+c_1 \alpha_{11} \alpha_{22}}{1+c_1 \alpha_{11}} & \frac{-c_1 \alpha_{13} \alpha_{21}+\alpha_{23}+c_1 \alpha_{11} \alpha_{23}}{1+c_1 \alpha_{11}} \\ t[3] & -\frac{\alpha_{31}}{(1+c_1 \alpha_{11})\,(1+c_1 \alpha_{11}+c_2 \alpha_{21}+c_3 \alpha_{31})} & \frac{-c_1 \alpha_{12} \alpha_{31}+\alpha_{32}+c_1 \alpha_{11} \alpha_{32}}{1+c_1 \alpha_{11}} & \frac{-c_1 \alpha_{13} \alpha_{31}+\alpha_{33}+c_1 \alpha_{11} \alpha_{33}}{1+c_1 \alpha_{11}} \end{pmatrix}$$

$$\begin{pmatrix} 0 & h[1] & h[2] & h[3] \\ t[1] & -\alpha_{11} & \alpha_{12} & \alpha_{13} \\ t[2] & -\alpha_{21} & \alpha_{22} & \alpha_{23} \\ t[3] & -\alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$