Pensieve Header: Computing the Alexander polynomial in β-calculus.

```
In[1]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\2012-01"];
<< betaCalculus.m
<< KnotTheory`
GC[K_] := GC @@ (
   PD[K] /. X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]],
      Ar[l, i, +1], Ar[j, i, -1]
     ]
  )
```

```
Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.
```

The Program and a Test Run

```
In[5]:= {K = Knot[8, 17];
 Alexander[K][X], GC[K]
}
```

KnotTheory::loading : Loading precomputed data in PD4Knots`.

Out[5]= $\left\{11-\frac{1}{X^3}+\frac{4}{X^2}-\frac{8}{X}-8X+4X^2-X^3\right.$, GC[Ar[1, 6, 1], Ar[7, 14, 1], Ar[3, 8, -1], Ar[13, 2, -1], Ar[5, 12, -1], Ar[9, 4, -1], Ar[11, 16, 1], Ar[15, 10, 1]]}

```
βForm[Plus @@ (GC[K] /. {Ar[i_, j_, +1] :> R[i, j], Ar[i_, j_, -1] :> RInv[i, j]})];
```

```
βAlex[K_] := Module[
  {gc, β},
  gc = GC[K];
  β =
   βCollect[Plus @@ (gc /. {Ar[i_, j_, +1] :> R[i, j], Ar[i_, j_, -1] :> RInv[i, j]})];
  Do[β = dm[1, k, 1][β], {k, 2, 2 Length[gc]}];
  Expand[β /. h[1] → 0 /. E^(n_. c[1]) :> X^n /. W[a_] :> a]
 ]
```

```
βAlex[K]
```

$$-8-\frac{1}{X^2}+\frac{4}{X}+11X-8X^2+4X^3-X^4$$

Testing the Full Rolfsen Table

```
{βAlex[#], Alexander[#][X]} & /@ AllKnots[{3, 7}] // MatrixForm
```

$$\begin{pmatrix}
X - X^2 + X^3 & -1 + \frac{1}{X} + X \\
-1 + 3X - X^2 & 3 - \frac{1}{X} - X \\
X - X^2 + X^3 - X^4 + X^5 & 1 + \frac{1}{X^2} - \frac{1}{X} - X + X^2 \\
2X^2 - 3X^3 + 2X^4 & -3 + \frac{2}{X} + 2X \\
-2X + 5X^2 - 2X^3 & 5 - \frac{2}{X} - 2X \\
-1 + 3X - 3X^2 + 3X^3 - X^4 & -3 - \frac{1}{X^2} + \frac{3}{X} + 3X - X^2 \\
5 + \frac{1}{X^2} - \frac{3}{X} - 3X + X^2 & 5 + \frac{1}{X^2} - \frac{3}{X} - 3X + X^2 \\
X - X^2 + X^3 - X^4 + X^5 - X^6 + X^7 & -1 + \frac{1}{X^3} - \frac{1}{X^2} + \frac{1}{X} + X - X^2 + X^3 \\
3X^3 - 5X^4 + 3X^5 & -5 + \frac{3}{X} + 3X \\
\frac{2}{X^5} - \frac{3}{X^4} + \frac{3}{X^3} - \frac{3}{X^2} + \frac{2}{X} & 3 + \frac{2}{X^2} - \frac{3}{X} - 3X + 2X^2 \\
\frac{4}{X^4} - \frac{7}{X^3} + \frac{4}{X^2} & -7 + \frac{4}{X} + 4X \\
2X^2 - 4X^3 + 5X^4 - 4X^5 + 2X^6 & 5 + \frac{2}{X^2} - \frac{4}{X} - 4X + 2X^2 \\
-1 + 5X - 7X^2 + 5X^3 - X^4 & -7 - \frac{1}{X^2} + \frac{5}{X} + 5X - X^2 \\
-5 + \frac{1}{X} + 9X - 5X^2 + X^3 & 9 + \frac{1}{X^2} - \frac{5}{X} - 5X + X^2
\end{pmatrix}$$

```
Test[K_] := Factor[βAlex[K] / Alexander[K][X]]
Union[Test /@ AllKnots[{3, 11}]]
```

KnotTheory::loading : Loading precomputed data in DTCode4KnotsTo11`.

KnotTheory::credits :
The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.

$$\left\{1, \frac{1}{X^6}, \frac{1}{X^5}, \frac{1}{X^4}, \frac{1}{X^3}, \frac{1}{X^2}, \frac{1}{X}, X, X^2, X^3, X^4, X^5, X^6\right\}$$

Avoiding Exponentials

```
ar[1, 2] ** (-1/(1 + c[1]) ar[1, 2])
```

0

```
βalex[K_] := Module[
  {gc, β},
  gc = GC[K];
  β = βCollect[W[1] + Plus @@
      (gc /. {Ar[i_, j_, +1] :> ar[i, j], Ar[i_, j_, -1] :> -1/(1 + c[i]) ar[i, j]})];
  Do[β = dm[1, k, 1][β], {k, 2, 2 Length[gc]}];
  Expand[β] /. c[1] → Y
 ]
```

```
βalex[K]
```

$$W\left[-\frac{-1-3Y-2Y^2+Y^3+3Y^4+2Y^5+Y^6}{(1+Y)^2}\right]$$

```
{Expand[βalex[#] /. {h[1] → 0, W[a_] ⧴ a} /. {Y → X - 1}],
    Expand[βalex[#] /. {h[1] → 0, W[a_] ⧴ a}], βAlex[#],
    Alexander[#][X], Conway[#][Y]} & /@ AllKnots[{3, 7}] // MatrixForm
```

$X-X^2+X^3$	$1+2Y+2Y^2+Y^3$	$X-X^2+X^3$
$-1+3X-X^2$	$1+Y-Y^2$	$-1+3X-X^2$
$X-X^2+X^3-X^4+X^5$	$1+3Y+6Y^2+7Y^3+4Y^4+Y^5$	$X-X^2+X^3-X^4+X^5$
$2X^2-3X^3+2X^4$	$1+3Y+5Y^2+5Y^3+2Y^4$	$2X^2-3X^3+2X^4$
$-2X+5X^2-2X^3$	$1+2Y-Y^2-2Y^3$	$-2X+5X^2-2X^3$
$-1+3X-3X^2+3X^3-X^4$	$1+2Y-Y^3-Y^4$	$-1+3X-3X^2+3X^3-X^4$
$5+\frac{1}{X^2}-\frac{3}{X}-3X+X^2$	$\frac{1}{(1+Y)^2}+\frac{2Y}{(1+Y)^2}+\frac{2Y^2}{(1+Y)^2}+\frac{Y^3}{(1+Y)^2}+\frac{Y^4}{(1+Y)^2}$	$5+\frac{1}{X^2}-\frac{3}{X}-3X+X^2$
$X-X^2+X^3-X^4+X^5-X^6+X^7$	$1+4Y+12Y^2+22Y^3+24Y^4+16Y^5+6Y^6+Y^7$	$X-X^2+X^3-X^4+X^5-X^6+X$
$3X^3-5X^4+3X^5$	$1+4Y+9Y^2+13Y^3+10Y^4+3Y^5$	$3X^3-5X^4+3X^5$
$\frac{2}{X^5}-\frac{3}{X^4}+\frac{3}{X^3}-\frac{3}{X^2}+\frac{2}{X}$	$\frac{1}{(1+Y)^5}+\frac{2Y}{(1+Y)^5}+\frac{6Y^2}{(1+Y)^5}+\frac{5Y^3}{(1+Y)^5}+\frac{2Y^4}{(1+Y)^5}$	$\frac{2}{X^5}-\frac{3}{X^4}+\frac{3}{X^3}-\frac{3}{X^2}+\frac{2}{X}$
$\frac{4}{X^4}-\frac{7}{X^3}+\frac{4}{X^2}$	$\frac{1}{(1+Y)^4}+\frac{Y}{(1+Y)^4}+\frac{4Y^2}{(1+Y)^4}$	$\frac{4}{X^4}-\frac{7}{X^3}+\frac{4}{X^2}$
$2X^2-4X^3+5X^4-4X^5+2X^6$	$1+4Y+10Y^2+16Y^3+15Y^4+8Y^5+2Y^6$	$2X^2-4X^3+5X^4-4X^5+2$
$-1+5X-7X^2+5X^3-X^4$	$1+2Y+2Y^2+Y^3-Y^4$	$-1+5X-7X^2+5X^3-X^4$
$-5+\frac{1}{X}+9X-5X^2+X^3$	$\frac{1}{1+Y}+\frac{2Y}{1+Y}-\frac{Y^3}{1+Y}+\frac{Y^4}{1+Y}$	$-5+\frac{1}{X}+9X-5X^2+X^3$

```
In[22]:= βSimplify = Factor;
gc = GC[K];
β = βCollect[W[1] +
    Plus @@ (gc /. {Ar[i_, j_, +1] ⧴ ar[i, j], Ar[i_, j_, -1] ⧴ -1/(1 + c[i]) ar[i, j]})];
(
  Table[{
    (β = dm[1, k, 1][β]) /. _c → c - 1 // βCollect // βForm,
    Collect[β /. {_W → 0, t[s_] ⧴ c[s]}, _h, Factor]
   }, {k, 2, 2 Length[gc]}]
 ) // ColumnForm
```

Out[25]=

W[1]	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	1	0	0	0	0	0
t[3]	0	0	0	$-\frac{1}{c}$	0	0	0	0
t[5]	0	0	0	0	0	$-\frac{1}{c}$	0	0
t[7]	0	0	0	0	0	0	1	0
t[9]	0	$-\frac{1}{c}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	0	0	0	0	0	0	0
t[15]	0	0	0	0	1	0	0	0

$$, -\frac{c[13]\,h[1]}{1+c[13]}-\frac{c[9]\,h[4]}{1+c[9]}+c[1]\,h[6]-\frac{c[}{1}$$

{

W[1]	h[1]	h[4]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	0	1	-1	0	0	0	0
t[5]	0	0	0	0	0	$-\frac{1}{c}$	0	0
t[7]	0	0	0	0	0	0	1	0
t[9]	0	$-\frac{1}{c}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	0	0	$\frac{-1+c}{c}$	0	0	0	0
t[15]	0	0	0	0	1	0	0	0

$, -\frac{c[13]\,h[1]}{1+c[13]} - \frac{c[9]\,h[4]}{1+c[9]} + c[1]\,h[6] - \frac{c[}{1}$

W[1]	h[1]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	1	-1	0	0	0	0
t[5]	0	0	0	0	$-\frac{1}{c}$	0	0
t[7]	0	0	0	0	0	1	0
t[9]	$-\frac{1}{c^2}$	0	0	0	0	0	0
t[11]	0	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	0	$\frac{-1+c}{c}$	0	0	0	0
t[15]	0	0	0	1	0	0	0

$, -\frac{(c[9]+c[13]+c[9]\,c[13])\,h[1]}{(1+c[9])\,(1+c[13])} + c[1]\,h[6] - \frac{c[1}{1+}$

W[1]	h[1]	h[6]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	0	1	-1	0	$-c$	0	0
t[7]	0	0	0	0	0	1	0
t[9]	$-\frac{1}{c^2}$	0	0	0	$\frac{-1+c}{c}$	0	0
t[11]	0	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	0	$\frac{-1+c}{c}$	0	$-1+c$	0	0
t[15]	0	0	0	1	0	0	0

$, -\frac{(c[9]+c[13]+c[9]\,c[13])\,h[1]}{(1+c[9])\,(1+c[13])} + c[1]\,h[6] - \frac{c[1}{1+}$

W[1]	h[1]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	$\frac{1}{c^2}$	-1	0	$-c$	0	0
t[7]	0	0	0	0	1	0
t[9]	$-\frac{1}{c^2}$	0	0	$\frac{-1+c}{c}$	0	0
t[11]	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	$\frac{-1+c}{c}$	0	$-1+c$	0	0
t[15]	0	0	1	0	0	0

$, \frac{(c[1]-c[9]-c[13]-c[9]\,c[13])\,h[1]}{(1+c[9])\,(1+c[13])} - \frac{c[1]\,h[8]}{1+c[1]} + c[15]\,h[$

W[1]	h[1]	h[8]	h[10]	h[12]	h[14]	h[16]
t[1]	$\frac{1}{c^2}$	-1	0	$-c$	$\frac{-1+c+c^2}{c}$	0
t[9]	$-\frac{1}{c^2}$	0	0	$\frac{-1+c}{c}$	$-\frac{-1+c}{c}$	0
t[11]	0	0	0	0	0	1
t[13]	$-\frac{1}{c}$	$\frac{-1+c}{c}$	0	$-1+c$	$1-c$	0
t[15]	0	0	1	0	0	0

$, \frac{(c[1]-c[9]-c[13]-c[9]\,c[13])\,h[1]}{(1+c[9])\,(1+c[13])} - \frac{c[1]\,h[8]}{1+c[1]} + c[15]\,h$

W[1]	h[1]	h[10]	h[12]	h[14]	h[16]
t[1]	$-\frac{-1+c}{c^2}$	0	$-c$	$\frac{-1+c+c^2}{c}$	0
t[9]	$-\frac{1}{c^2}$	0	$\frac{-1+c}{c}$	$-\frac{-1+c}{c}$	0
t[11]	0	0	0	0	1
t[13]	$-\frac{1}{c^2}$	0	$-1+c$	$1-c$	0
t[15]	0	1	0	0	0

$, -\frac{(c[9]+c[13]+c[9]\,c[13])\,h[1]}{(1+c[9])\,(1+c[13])} + c[15]\,h[10] - \frac{c[1]\,h[12]}{1+c[1]} + c$

$$\left\{\begin{pmatrix} W[c] & h[1] & h[10] & h[12] & h[14] & h[16] \\ t[1] & -\frac{-1+c+c^2}{c^3} & 0 & -\frac{-1+3c-2c^2+c^3}{c^2} & \frac{-1+2c-c^2+c^3}{c^2} & 0 \\ t[11] & 0 & 0 & 0 & 0 & 1 \\ t[13] & -\frac{1}{c^3} & 0 & \frac{(-1+c)\left(1-c+c^2\right)}{c^2} & -\frac{(-1+c)\left(1-c+c^2\right)}{c^2} & 0 \\ t[15] & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, -\frac{(c[1]+c[13]+c[1]\,c[13])\,h[1]}{(1+c[1])\,(1+c[13])}+c[15]\,h[\right.$$

$$\left\{\begin{pmatrix} W[c] & h[1] & h[12] & h[14] & h[16] \\ t[1] & -\frac{-1+c+c^2}{c^3} & -\frac{-1+3c-2c^2+c^3}{c^2} & \frac{-1+2c-c^2+c^3}{c^2} & 0 \\ t[11] & 0 & 0 & 0 & 1 \\ t[13] & -\frac{1}{c^3} & \frac{(-1+c)\left(1-c+c^2\right)}{c^2} & -\frac{(-1+c)\left(1-c+c^2\right)}{c^2} & 0 \\ t[15] & \frac{1}{c^2} & 0 & 0 & 0 \end{pmatrix}, -\frac{(c[1]+c[13]+c[1]\,c[13]-c[15])\,h[1]}{(1+c[1])\,(1+c[13])}-\frac{c[1]\,h[12]}{1+c[1]}+(\right.$$

$$\left\{\begin{pmatrix} W[c] & h[1] & h[12] & h[14] & h[16] \\ t[1] & -\frac{-1+c+c^2}{c^3} & -\frac{-1+3c-2c^2+c^3}{c^2} & \frac{-1+2c-c^2+c^3}{c^2} & \frac{-1+2c}{c^2} \\ t[13] & -\frac{1}{c^3} & \frac{(-1+c)\left(1-c+c^2\right)}{c^2} & -\frac{(-1+c)\left(1-c+c^2\right)}{c^2} & -\frac{-1+c}{c^2} \\ t[15] & \frac{1}{c^2} & 0 & 0 & \frac{-1+c}{c} \end{pmatrix}, -\frac{(c[1]+c[13]+c[1]\,c[13]-c[15])\,h[1]}{(1+c[1])\,(1+c[13])}-\frac{c[1]\,h[12]}{1+c[1]}+(\right.$$

$$\left\{\begin{pmatrix} W[c] & h[1] & h[14] & h[16] \\ t[1] & -\frac{-2+4c-c^2+c^3}{c^3} & \frac{-1+2c-c^2+c^3}{c^2} & \frac{-1+2c}{c^2} \\ t[13] & \frac{-2+2c-2c^2+c^3}{c^3} & -\frac{(-1+c)\left(1-c+c^2\right)}{c^2} & -\frac{-1+c}{c^2} \\ t[15] & \frac{1}{c^2} & 0 & \frac{-1+c}{c} \end{pmatrix}, -\frac{\left(2c[1]+c[1]^2+c[13]+2c[1]\,c[13]+c[1]^2\,c[13]-c[15]\right)h[1]}{(1+c[1])^2\,(1+c[13])}+c[1]\right.$$

$$\left\{\begin{pmatrix} W\left[-(-2+c)\left(-1+2c-c^2+c^3\right)\right] & h[1] & h[14] & h[16] \\ t[1] & -\frac{2-4c-c^2+c^3-2c^4+c^5}{(-2+c)\,c^2\left(-1+2c-c^2+c^3\right)} & \frac{-1+5c-9c^2+7c^3-4c^4+c^5}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} & \frac{1-3c+3c^2-3c^3+c^4}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} \\ t[15] & -\frac{1}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} & \frac{(-1+c)^2\left(1-c+c^2\right)}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} & \frac{(-1+c)\left(1-4c+4c^2-3c^3+c^4\right)}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} \end{pmatrix},\right.$$

$$\left\{\begin{pmatrix} W\left[-(-2+c)\left(-1+2c-c^2+c^3\right)\right] & h[1] & h[16] \\ t[1] & -\frac{1-3c+5c^2-8c^3+5c^4-3c^5+c^6}{(-2+c)\,c^3\left(-1+2c-c^2+c^3\right)} & \frac{1-3c+3c^2-3c^3+c^4}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} \\ t[15] & \frac{1-3c+3c^2-3c^3+c^4}{(-2+c)\,c^3\left(-1+2c-c^2+c^3\right)} & \frac{(-1+c)\left(1-4c+4c^2-3c^3+c^4\right)}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)} \end{pmatrix}, -\frac{\left(2c[1]+c[1]^2-c[15]\right.}{(1+c[1])^2}\right.$$

$$\left\{\begin{pmatrix} W\left[-\frac{1-4c+8c^2-11c^3+8c^4-4c^5+c^6}{c^2}\right] & h[1] & h[16] \\ t[1] & -\frac{1}{c} & 1 \end{pmatrix}, -\frac{c[1]\,h[1]}{1+c[1]}+c[1]\,h[16]\right\}$$

$$\left\{\left(W\left[-\frac{1-4c+8c^2-11c^3+8c^4-4c^5+c^6}{c^2}\right]\right), 0\right\}$$

$-\frac{2-4c-c^2+c^3-2c^4+c^5}{(-2+c)\,c^2\left(-1+2c-c^2+c^3\right)}+-\frac{1}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)}$ // Simplify

$$-\frac{1+c}{c^2}$$

$\frac{-1+5c-9c^2+7c^3-4c^4+c^5}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)}+\frac{(-1+c)^2\left(1-c+c^2\right)}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)}$ // Simplify

1

$\frac{1-3c+3c^2-3c^3+c^4}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)}+\frac{(-1+c)\left(1-4c+4c^2-3c^3+c^4\right)}{(-2+c)\,c\left(-1+2c-c^2+c^3\right)}$ // Simplify

1