Tolman: Toric Integrable Geodesic Flows

joint w/ Chris Lee.

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* Q: compact n-dim connected manifold.
* g: Riemannian metric
* T^*Q: The cotangent bundle.
* Ω: the canonical sympl. form on T^*Q

$$\omega = \sum dq_i \wedge dp_i \qquad \begin{array}{l} q \text{ on } Q \\ p \in T^*Q \end{array}$$

* $h: T^*Q \to \mathbb{R}$ via

$$h(q,p) = \sqrt{g_q^*(p,p)}$$

* The geodesic flow is the flow of the Hamiltonian v.f. of h.

The geodesic flow is "toric integrable" if there is an effective action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ on $T^*Q \setminus Q$

* preserves Ω
* commutes w/ dilations
* commutes w/ geodesic flow (or equiv, preserves h)

Example: 1. Flat metric on $\mathbb{T}^n$

2. Round metrics on $S^2, \mathbb{RP}^2, S^3/\mathbb{Z}_l$

Toth + Zelditch Examined connects between the dynamics of g. flow & eigs of the Laplace operator.

Thm (Lerner - Shirokova, following conj. by T&Z) Every toric integrable geodesic flow on the torus is flat.

But (Verdière) not every such flow on S^2 is round.

Question (Lerman) are the above examples the only manifolds that admit toric integrable geodesic flows.

Thm A. (Lee, Tolman) If the g. flow is toric integrable, and either n is odd or $\pi_1(Q)$ is infinite then Q is either T^n or $S^3/\mathbb{Z}_\ell$

Idea of proof. $\alpha = \sum p_i dq_i$ "Liouville form"

$\alpha|_{S(T^*Q)}$ is a contact form

$S(T^*Q)$ preserved by action

$\Rightarrow S(T^*Q)$ is a contact toric mfld.

now comes...

Thm B. Assumptions as before but $n \neq 3$, then $S(T^*Q) \underset{\substack{\uparrow \\ \text{equivariant contactomorphic}}}{\cong} S(T^*\mathbb{T}^n)$

Thm B $\Rightarrow$ Thm A : .... $\pi_1(Q) = \mathbb{Z}^n$

$\pi_i(Q) = 0 \quad i > 1$

by whitehead Q is homotopy equiv. to $\mathbb{T}^n$;

and by Hsiang-Wall $(n \geq 5)$

Freedman $(n=4)$

Perelman $(n=3)$

we get a homeomorphism. 2:40