Coleman Lecture 5: Symmetries and Conservation Laws

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$$\text{quatidn.}\ \phi^a = d\phi^a$$

$$L = L(q^1, \ldots q^n, \dot{q}^1, \ldots, \dot{q}^n, t)$$

Symmetry:

$$q^a \mapsto q^a(t, \lambda) \quad \text{s.t.} \quad q^a(0, \lambda) = q^a$$

E.g. $$L = \frac{1}{2} m_0 \dot{\vec{x}} \cdot \dot{\vec{x}} + \sum V_{ij}(x_i - x_j)$$

symmetric under translations.

Infinitesimally,

$$q^a \mapsto q^a + Dq^a \cdot d\lambda \qquad Dq^a = \ldots$$

$$D\dot{q}^a = \frac{d}{dt} Dq^a$$

$$DL = \frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a$$

A transformation is a symmetry if

$$DL = dF/dt \qquad F = F(q, \dot{q}, t)$$

Infinitesimal Sym $\Rightarrow$ Conservation Laws:

$$Q = p_a Dq^a - F$$ Q for "quantity"

$$\frac{dQ}{dt} = \dot{p}_a Dq^a + p_a D\dot{q}^a - dF/dt$$

$$\frac{dQ}{dt} = \dot{p}_a Dq^a + p_a D\dot{q}^a - dF/dt$$

$$= \underbrace{\frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a}_{DL} - \frac{dF}{dt} = 0$$

Examples in particle mechanics:

1. $x^{(r)} \mapsto x^{(r)} + \lambda \vec{e}$ in the above example.

$$Dx^{(r)} = e \qquad F = 0$$

$$Q = \sum_r m_r \dot{x}^{(r)} \cdot \vec{e}$$

2. Assume $\frac{\partial L}{\partial t} = 0$ $\quad q^a(t) \to q^a(t+\lambda)$

$$Dq^a = \dot{q}^a$$

$$DL = \frac{dL}{dt} \qquad F = L$$

$$Q = p_a \dot{q}^a - F = p_a \dot{q}^a - L$$

"energy"
"Hamiltonian"

why didn't I get coffee?

In quantum mechanics,

$$[q^a, Q] = i Dq^a$$

Classical Field Theory

$$\mathcal{L}(\phi^1, \ldots, \phi^a, \partial_\mu \phi^1, \ldots \partial_\mu \phi^n, x^\mu)$$

$\phi^a(x) \mapsto \phi^a(x,\lambda) \qquad \phi(x,0) = \phi(x)$

"Symmetry" if $D\mathcal{L} = \partial_\mu F^\mu$

F of prev. time

$DL = \partial_0 \int d^3x\, F^0 \qquad F = \int d^3x\, F^0$

$$D\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^a} D\phi^a + \Pi_\mu{}^a \partial_\mu D\phi^a$$

$$J^\mu = \Pi^\mu_a D\phi^a - F^\mu$$

$$\partial_\mu J^\mu = \underbrace{(\partial_\mu \Pi^\mu_a)}_{\partial\mathcal{L}/\partial\phi^a} D\phi^a + \Pi^\mu_a \partial_\mu D\phi^a - \partial_\mu F^\mu$$

$= 0$

"a conserved current"

J^0 : "density"

$\vec{J}$: "current" get $\partial_0 J^0 + \nabla \cdot J = 0$

$$Q = \int d^3\vec{x}\, J^0(\vec{x}) \qquad \frac{dQ}{dt} = 0$$

under $F^\mu \longrightarrow F^\mu + \partial_\nu A^{\mu\nu}$ $\quad A^{\mu\nu}$ anti-symmetric

the new F is as good, yet

$$J^\mu \longrightarrow J^\mu + \partial_\nu A^{\mu\nu}$$

... yet the overall Q doesn't change.

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Space Translation.

$$\phi^a(x) \mapsto \phi^a(x+\lambda e) \qquad e \in \mathbb{R}^4$$

$$D\phi^a = e_\rho \partial^\rho \phi^a$$

Expect $\quad J^\mu = e_\rho T^{\rho\mu}$ $\quad T^{\rho\mu}$ is "the canonical energy momentum tensor"

$$p^\rho = \int d^3x\, T^{\rho 0} \qquad \text{Energy-momentum}$$

$$D\mathcal{L} = e_\rho \partial^\rho \mathcal{L} = \partial_\mu g^{\mu\rho} e_\rho \mathcal{L} = \partial_\mu F^\mu$$

$$J^\mu = \pi^\mu_a e_\rho \partial^\rho \phi^a - g_{..} e_\rho \mathcal{L}$$

$$T^{\rho\mu} = \pi^\mu_a \partial^\rho \phi^a - g^{\rho\mu}\mathcal{L}$$

$$T^{00} = \pi^0_a \partial^0 \phi^a - \mathcal{L}$$ the energy density = the Hamiltonian density

Example:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\mu^2}{2}\phi^2$$

$$\pi^\mu = \partial^\mu\phi \qquad T^{i0} = (\partial^0\phi)(\partial^i\phi)$$

$$P = \int d^3x \left(-\partial^0\phi \vec{\nabla}\phi\right)$$

Plugging in $\phi = \int \frac{d^3k}{\sim\ ..} \left[\begin{array}{l}\text{creation} \\ \& \text{ annihilation}\end{array}\right]$

... get

$$\frac{1}{2}\int d^3k\,[a_k a_k^* + a_k^* a_k]\,\vec{k}$$

$$= \frac{1}{2}\int d^3k\,[a_k a_k^*] \quad \mathcal{2}_0$$

— ... video rattles ...

Lorentz Transformations.

$$x^\mu \mapsto x^\mu + \epsilon^{\mu\nu} x_\nu\, d\lambda$$

is a Lorentz transformation iff

$$\underline{\epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0 \iff \epsilon^{\mu\nu} + \epsilon^{\nu\mu} = 0}$$

Example $\epsilon^{12} = -\epsilon^{21} = 1$, all others $= 0$

$\Rightarrow$ rotation about z-axis.

$\epsilon^{10} = 1 = -\epsilon^{01}$, all other $= 0$

$\Rightarrow$ Pure Lorentz in x direction.

$$J^\mu = \frac{1}{2}\epsilon_{\lambda\rho} M^{\lambda\rho\mu} \qquad \partial_\mu M^{\lambda\rho\mu} = 0$$

$\Rightarrow$ get 3 conserved angular momenta

and 3 "Lorentzian conservation laws"

$$\phi^a(x) \rightarrow \phi^a(\Lambda x)$$

$$(\Lambda x)^\mu = x^\mu + \epsilon^{\mu\nu} x_\nu \cdot \lambda$$

$$D\phi^a = \epsilon^{\rho\sigma} x_\sigma \partial_\rho \phi^a \ldots$$

$$D\mathcal{L} = \ldots$$

$$J_\mu = \epsilon_{\rho\sigma}\left(x^\sigma \ldots \right)$$

$$= x^\sigma T_{\rho\mu}$$ can be anti-symmetrized

to give

$$M^{\lambda\rho\mu} = x^\lambda T^{\rho\mu} - x^\rho T^{\lambda\mu}$$