Coleman Lecture 4: CPM, QM, FT, QFT; normal ordering

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$$\begin{array}{ccccc} 1 & \boxed{\text{CPM}} & — & \boxed{\text{QM}} & 2 \\ & | & & | & \\ 3 & \boxed{\text{FT}} & — & \boxed{\text{QFT}} & 4 \end{array}$$

Box 1 –

Classical mechanics: $q^a(t)$

The Lagrangian $L_0(q^1 \ldots q^n, \dot{q}^1, \ldots \dot{q}^n, t)$

$$S = \int_{t_1}^{t_2} dt\, L_0$$

Hamilton's principles $q^a \mapsto q^a + \delta q^a$ $\delta q^a(t_1, t_2) = 0$

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial q^a} - \frac{d}{dt} \underbrace{\frac{\partial L}{\partial \dot{q}^a}}_{p_a} = 0$$

$$p_a \dot{q}^a - F = H(q^a, p_a, t) = \left(\begin{matrix} \text{Energy if} \\ L \text{ is time} \\ \text{indep} \end{matrix} \right)$$

$$\frac{\partial H}{\partial p_a} = \dot{q}^a$$

$$\frac{\partial H}{\partial q^a} = -\dot{p}_a$$

The Hamilton equations of motion.

The p's & q's must be "complete" and "independent".

<u>Example</u> A particle of unit mass constrained to move on a sphere:

$$L = \frac{1}{2}\dot{\vec{x}}^2 + \lambda(\vec{x}^2 - 1)$$

$$L = \tfrac{1}{2}x^{\cdot} + \lambda(x-1)$$

$$p_\lambda = 0 \qquad \text{"not an independent variables"}$$

Box 2. Canonical quantization

$$[p^a(t), q_b(t)] = i\delta^a_b$$

The Hamiltonian becomes an operator ... which is ambigously-defined.

$$\ldots \quad [q^a, H] = i\frac{\partial H}{\partial p_a}$$

$$\dot{q}^a = -i[q^a, H] \qquad \dot{p}_a = -i[p_a, H]$$

Box 3. Classical field theory.

$$q^a(t) \longmapsto \phi^a(\vec{x}, t)$$

$$a \Longleftrightarrow (a, \vec{x})$$

specialize to the case

$$L = \int d^3\vec{x}\, \mathcal{L}(\phi^a(\vec{x}), \partial_\mu\phi^a(\vec{x}), \ldots)$$

$$S = \int_{t_1}^{t_2} L = \int_{t_1<t<t_2} d^4x \cdot \mathcal{L}$$

$$\delta S = \int d^4x \left(\frac{\partial\mathcal{L} \ldots}{\partial(\partial_\mu\phi^a)} + \ldots\right.$$

$$\Pi^{\mu}_{a} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}$$

$$\Rightarrow \quad \partial_\mu \Pi^{\mu}_{a} - \frac{\partial \mathcal{L}}{\partial \phi^a} = 0$$

Example One scalar field $\phi(x)$, quadratic L, Lorentz invariant.

$$\mathcal{L} = \pm\frac{1}{2}\left[\partial_\mu\phi\, \partial^\mu\phi + \frac{b}{a}\phi^2\right] \qquad \Pi_\mu = \frac{\partial \mathcal{L}}{\partial_\mu \phi} = \pm\partial^\mu\phi$$

$$\partial_\mu \Pi^\mu \mp \frac{b}{a}\phi = 0 \qquad \Box^2\phi \mp \frac{b}{a}\phi = 0$$

$$\delta L = \delta\int d^3x\, \mathcal{L} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0\phi^a)}(\;\;\cdot\;\;) + \begin{matrix}\text{no}\\\text{time}\\\text{deriv.}\end{matrix}$$

$$\Pi^0_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^a)}$$

$$H = \int d^3x\left(\Pi^0_a\dot{\phi}^a - \mathcal{L}\right)$$
$$= \int d^3x\, \mathcal{H} \quad \leftarrow \text{The Hamiltonian density.}$$

In the example, $\Pi = \pm\dot{\phi}$

$$\mathcal{H} = \pm\left[\Pi\dot{\phi} - \ldots\right]$$
$$= \pm\left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\text{grad}\,\phi)^2 - \frac{b}{a}\phi^2\right)$$

$\Rightarrow$ to get $H > 0$ must take the $+$ choice, with $\frac{b}{a} < 0$. Write $-\frac{b}{a} = \mu^2$ & get

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\text{grad}\,\phi)^2 + \mu^2\phi^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi - \mu^2\phi^2)$$

$$(\Box^2 - \mu^2)\phi = 0$$

Box 4. Quantum Field theory

$$[\phi^a(x,t), \pi_b(y,t)] = i\delta^a_b\,\delta^{(3)}(x-y)$$

continuing with the example,

$$H = \frac{1}{2}\int d^3y\,\left(\pi(y,t)^2 + \frac{1}{2}(\nabla_y\phi)^2 + \mu^2\phi^2(y,t)\right)$$

The Heisenberg eqn of motion:

$$\dot\phi(x,t) = -i[\phi(x,t), H] = \ldots =$$

$$= \pi(x,t)$$

$$\dot\pi(x,t) = -i[\pi(x,t), H] = \ldots =$$

$$= \nabla^2\phi(x,t) + \mu^2\phi$$

$$\ldots \quad \ddot\phi = \nabla^2\phi \pm \mu^2\phi$$

This was Chapter 11-12 of Bjorken-Drell. 1:17:52

October 11:

$\phi(\vec{x}, t) =$ (in terms of a, a^*)

Substitute into H, do huge calculation, get the "infinitely many Harmonic oscillators lagrangians" with wrong 0-energy. This is solved using "normal ordering": http://en.wikipedia.org/wiki/Normal_ordering

So the "correct" Hamiltonian is:

$$H = \frac{1}{2}\int d^3y : \pi(y,t)^2 + \frac{1}{2}(\nabla_y \phi)^2 + \mu^2 \phi^2(y,t) :$$