Gekhtman: Compatible Poisson structures in cluster algebras

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Cluster Algebras.

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Ingredients: 1. Initial cluster $X = \{\underbrace{x_1 \dots x_n}_{\text{cluster variables}}, \underbrace{x_{n+1}, \dots, x_{n+m}}_{\text{"stable", "frozen", "tropical" variables}}\}$

2 an integral $n \times (n+m)$ matrix:

$$B = \Big[\underbrace{B_0}_{n} \;\Big|\; B_1\Big]\}n$$

s.t. B_0 is skew-symmetrizable: $\exists$ diagonal integral D st. DB_0 is skew-symmetric.

3. cluster transformation: fix $1 \le i \le n$ and define

a. $$T_i(x_j) = \begin{cases} x_j & i \neq j \\ x_i' = \frac{1}{x_i}\left(\prod_{b_{ij} > 0} x_j^{b_{ij}} + \prod_{b_{ij} \le 0} x_j^{-b_{ij}}\right) & \text{otherwise} \end{cases}$$

b. $T_i(B) = \tilde{B}$ w/

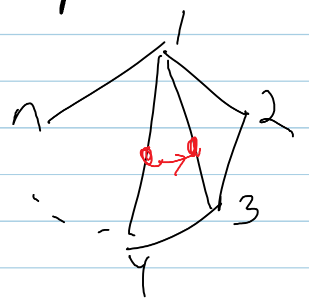
$$\tilde{b}_{jk} = \begin{cases} -b_{jk} & \text{if } i = j \text{ or } i = k \\ b_{jk} + \frac{|b_{ji}| b_{ik} + b_{ji} |b_{ik}|}{2} & \text{otherwise} \end{cases}$$

Thm (Fomin-Zelevinsky)

All cluster variables [obtained by iterating

The above] are Laurant poly in X.

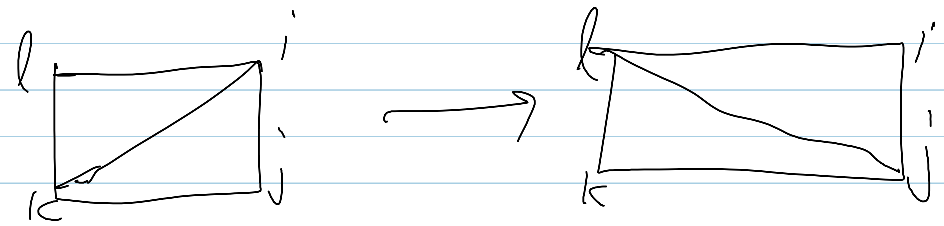
Example. $Gr(2,n)$



Fix a triangulation of the Stasheff polyhedron.

$x_{12}, \ldots x_{n1}$ } sides are the stable variables

Rules:



$$x_{ik} \longmapsto x_{jl} = \frac{x_{ij}x_{kl} + x_{li}x_{jk}}{x_{ik}}$$

x_{ij}: "plucker coordinates"

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Compatible Poisson Structures:

Assume B is of full rank & we have a poisson structure s.t.

$$\{x_i, x_j\} = c_{ij}\, x_i x_j$$

Say that $\{,\}$ is compatible with A if it has the same form in every cluster.

Claim Compatible $\{,\}$ exist and $\Omega = (c_{ij})$ is given by

$$\Omega B = [D_n \quad \circledast]$$

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