Alekseev @ Colloq: Eigenvalues, interlacing inequalities and tropical calculus

October-12-11
1:16 PM

1. Gelfand-Zeitlin map:

$\mathcal{H}_n = \{n \times n \text{ Hermitian matrices}\} \ni a$

$\begin{pmatrix} a_{1n} & a_{1n} \\ a_{n1} & \cdot a_{nn} \end{pmatrix}$

$$a \xmapsto{gz} \begin{pmatrix} \lambda_1^{(n)} \geq \ldots\ldots \geq \lambda_n^{(n)} \\ \lambda_1^{(n-1)} \geq \quad \geq \lambda_{n-1}^{(n-1)} \\ \vdots \\ \lambda_1^{(1)} \end{pmatrix}$$

The eigvals of a

eigvals of matrices obtained from a by removing the first few rows/columns.

total #:
$N = \frac{n(n+1)}{2}$

Interlacing inequalities:

$$\left.\lambda_i^{(k)} \geq \lambda_i^{(k-1)} \geq \lambda_{i+1}^{(k)}\right\} \quad C_{gz}\text{: The cone formed by these ineqs.}$$

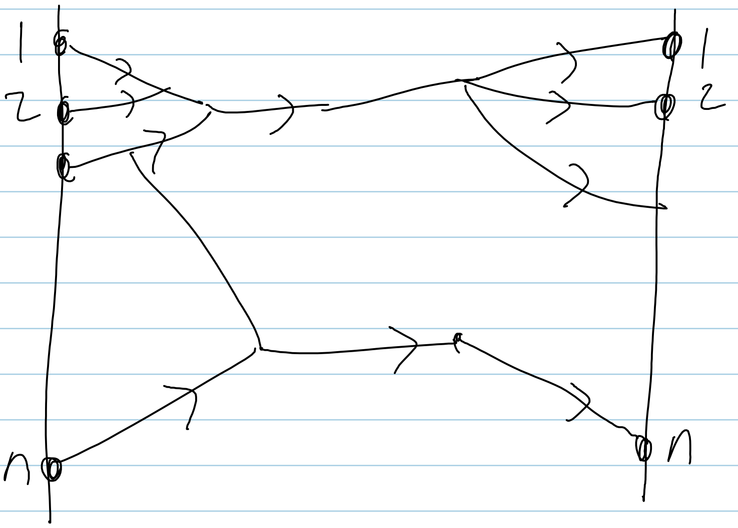
There's a proof using the min-max principle for eigenvalues.

<u>Fact:</u> $\operatorname{im}(gz) = C_{gz}$

$l_i^{(k)} = \lambda_1^{(k)} + \ldots + \lambda_i^{(k)} \qquad l_0^{(k)} = 0$

$$\left.\begin{aligned} l_i^{(k)} + l_{i+1}^{(k-1)} &\leq l_{i+1}^{(k)} + l_i^{(k-1)} \\ l_{i+1}^{(k)} + l_{i-1}^{(k-1)} &\leq l_i^{(k)} + l_i^{(k-1)} \end{aligned}\right\} \text{ equiv. ineq.}$$

Planar networks / total positivity



left to right planar graphs w/ weights on edges.

$$\{E\} \to \underbrace{\mathbb{R} \cup \{-\infty\}}_{\mathbb{T}}$$

Define $L : \mathbb{T}^{|E|} \longrightarrow \mathbb{T}^{N}$, $N = \frac{n(n+1)}{2}$

$$l_1^{(k)} = \max_{\substack{\text{path}: s \to s \\ s \in \{n-k+1, \ldots, n\}}} \left(\sum_{\text{path}} \text{weights} \right)$$

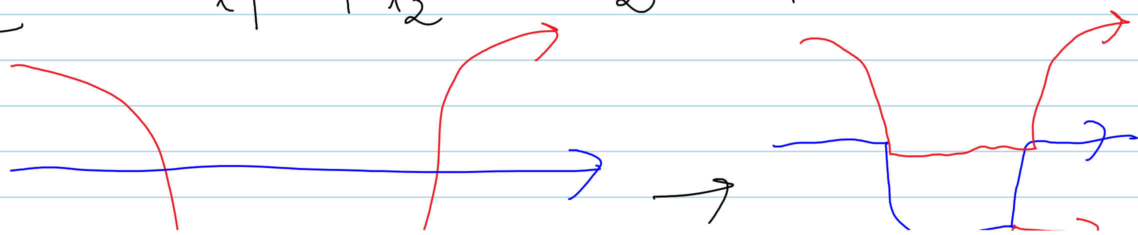
The empty path is $-\infty$.

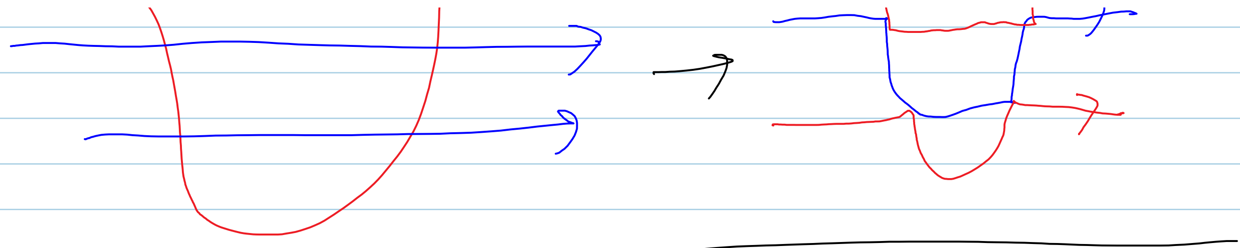
<u>Def</u> An i-path is a collection of i paths not touching each other.

$$l_i^{(k)} = \max_{\substack{i\text{-paths}: S \to S \\ S \subset \{n-k+1, \ldots, n\}}} \left(\sum_{i\text{-path}} \text{weights} \right)$$

<u>Thm</u> $\operatorname{im}(L) \subset C_{gz}$

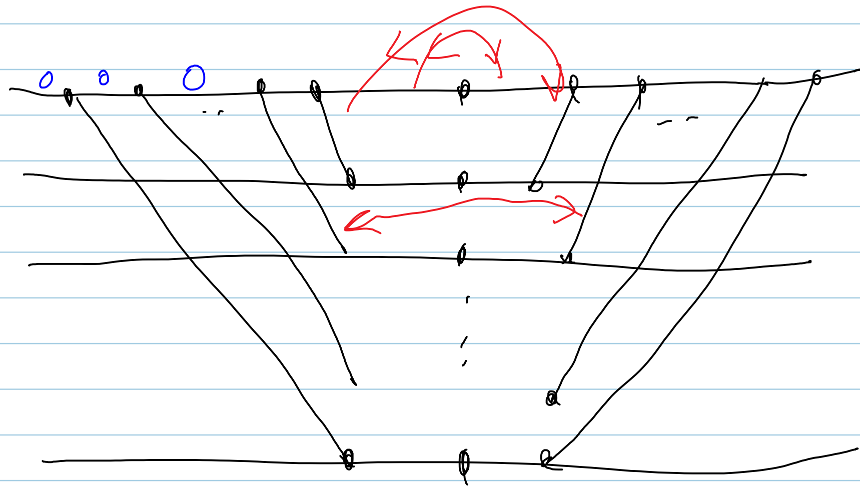
<u>Proof</u> $l_1^{(k)} + l_2^{(k-1)} \leq l_2^{(k)} + l_1^{(k-1)}$





Choice of network:

$$N = \frac{n(n+1)}{2}$$



$$L : \mathbb{T}^N \longrightarrow \mathbb{T}^N$$

<u>Thm</u> For the above graph,

* $\exists !$ linearity chamber C_0 s.t.

$$\mathrm{Jac}(L) \neq 0$$

* $L|_{C_0} \longrightarrow C_{az}$ is 1-1, all other chambers go to the ∂C_{gz}

3:48