Leclerc: Quantum Cluster Algebras and Quantum Groups

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5:04 AM

1. Quantum Groups.

$\mathfrak{g} = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_-$ semisimple Lie Alg. / ADE
Kac-Moody Lie Alg
Symmetric.

W: Weyl group.

Drinfeld-Jimbo: $U_q(\mathfrak{g})$: algebra over $\mathbb{C}(q)$

$$U_q(\mathfrak{g}) \supset U_q(\mathfrak{n}) \supset U_q(\mathfrak{n}(w))$$

$w \in W$
Lustig, Kac,
De Concini, Procesi

$$\mathfrak{n}(w) = \bigoplus_{\beta \in S_w} \mathfrak{n}_\beta \qquad S_w = \left\{ \begin{array}{l} \beta \in \Delta_+ \\ w(\beta) \in \Delta_- \end{array} \right\}$$

$w = s_{i_1} \cdots s_{i_k}$ (reduced expression for w)

$$S_w = \{ \underbrace{\alpha_{i_1}}_{\beta_1}, \underbrace{s_{i_1}(\alpha_{i_2})}_{\beta_2}, \ldots \underbrace{s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})}_{\beta_k} \}$$

<u>Aim</u> Describe $U_q(\mathfrak{n}(w))$ as a quantum cluster algebra.

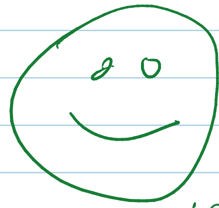
2. Quantum Cluster Algebras. (Berenstein-Zelevinsky; Fock-Goncharov)

quantum

Roughly $\dfrac{\mathbb{Q}(q)\langle x_1 \ldots x_n \rangle}{x_i x_j = q^{\lambda_{ij}} x_j x_i}$ } quantum affine n-space

λ_{ij} anti-symmetric

take another: $\dfrac{\mathbb{Q}(q)\langle y_1 \ldots y_n \rangle}{\ldots \mu_{ij} \ldots}$



why not?

fix k and identify $x_i = y_i$

$x_k y_k = m_1 + m_2$ } monomials in the remaining variables

Aim: $\left\{ \begin{array}{c}\text{quantum cluster} \\ \text{monomials}\end{array} \right\} \subset B^*$ dual canonical basis (Lusztig)

Precise definition. Quantum seed:

$(Q\ ,\quad \Lambda = (\lambda_{ij})\ ,\quad (x_1 \ldots x_n))$

Q: quiver w/ n vertices, no loops, no 2-cycles

Λ: $n \times n$ matrix in $\mathbb{Z}$, anti-symmetric

$(x_1 \ldots x_n)$: generate a quantum n-torus:

$$\mathcal{T} = \frac{\mathbb{Q}(q)\langle x_1 \ldots x_n \rangle}{x_i x_j = q^{\lambda_{ij}} x_j x_i}$$

(Some compatibility between Λ & Q is assumed.)

OBN: There must be a relationship between the different bases of $A^{\omega}(\Gamma)$, corresp. to different maximal trees, and cluster algebras.