Cliff and Halacheva: Classification of Lie BiAlgebras

August-23-11
2:19 PM

Def. A Lie bialgebra structure on a Lie algebra $\mathfrak{g}$,

Examples. 1. $\delta_{\mathfrak{g}} = 0$

2. 2D: $\left.\begin{array}{l}[x,y]=x, \\ \delta x = \alpha(x\wedge y) \\ \delta y = \beta(x \wedge y)\end{array}\right\}$ after rescaling, either $\alpha = 0 = \beta$ or $\alpha=1, \beta=0$ or $\alpha=0, \beta=1$.

Def. A homomorphism of Lie bialgebras.

Def. Poisson manifolds, Poisson Lie groups, the relation w/ Lie bialgebras.

Prop. (Manin triples)

$$\left\{\begin{array}{l}\text{Lie bialg} \\ \text{structures on } \mathfrak{g}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{triples } (\mathfrak{p}_+, \mathfrak{p}_-, \mathfrak{p}) \\ \text{s.t. } \mathfrak{p}_+ = \mathfrak{g}\end{array}\right\}$$

Def. "The standard structure" on $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$

Def. Co-boundary Lie bialgebras – "δ is a 1-coboundary"; precisely, $\exists\, r \in \mathfrak{g}\otimes\mathfrak{g}$ s.t.

$$\delta(x) = x.r$$

Prop. If $\mathfrak{g}$ is a Lie algebra and $r \in \mathfrak{g}\otimes\mathfrak{g}$, then $\delta(x) := x.r$ is a bialgebra structure iff 1. $r_{12}+r_{21} \in (\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$

2. $[\![r,r]\!] := [r_{12},r_{13}]+[r_{12},r_{23}] + [r_{13},r_{23}] \in (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$

Example. The double of a Lie bialgebra is a coboundary Lie bialgebra, $\mathcal{D}(\mathfrak{g})$.

Def. A solution to the CYBE equation $[\![r,r]\!]=0$ is called "an r-matrix".

If $r+r_{21}$ is $\mathfrak{g}$-invariant it gives a "quasi-triangular coboundary structure"

If $r+r_{21}=0 \longrightarrow$ "triangular".

Do these things have a geometric meaning in the Poisson-Lie group case?

Examples. In 2D,

$\delta(x)=0,\ \delta(y)=x\wedge y$ coboundary w/ $r=x\wedge y$.

Theorem (Whitehead). If $\mathfrak{g}$ is a f.d. simple/$\mathbb{C}$ Lie algebra and M a f.g. $\mathfrak{g}$-module then

$$H^1(\mathfrak{g},M)=0=H^2(\mathfrak{g},M)$$

Taking $M=\mathfrak{g}\otimes\mathfrak{g}$ we find that every Lie bialg. on $\mathfrak{g}$ is co-boundary.

Proposition w/ $\mathfrak{g}$ as above, $(\cdot,\cdot)$,

$$(\Lambda^3 \mathfrak{g})^{\mathfrak{g}} = \mathbb{C}[\Omega_{13}, \Omega_{23}] \quad \Omega: \text{the Casimir.}$$

Proposition. If $\mathfrak{g}$ as above is coboundary,

⋮

Enough to classify r-matrices r s.t.

$$r + r_{21} = 0 \quad \text{or} \quad r + r_{21} = 1 \cdot \Omega$$

Thm $\left\{ \begin{matrix} \text{triangular (skew symmetric)} \\ r \text{ for } \mathfrak{g} \end{matrix} \right\}$

$$\longleftrightarrow \left\{ \begin{matrix} \text{Pairs} \\ (\mathfrak{h}, r^-) \ldots \end{matrix} \right\}$$

The non-skew-symmetric case: $r_{12} + r_{21} = \Omega$,

$\mathfrak{g}$ simple & f.d.

Def. A Belavin-Drinfel'd triple $(\Gamma_1, \Gamma_2, \tau)$ is

1. $\Gamma_1, \Gamma_2 \subset \{\text{simple roots}\}$
2. τ is a bijection that preserves adjacency in the Dynkin diagram.
3. $\forall \alpha \in \Gamma_1$, $\exists k \in \mathbb{N}$ s.t.

   $\alpha, \tau\alpha, \ldots, \tau^{k-1}(\alpha) \in \Gamma_1$, yet $\tau^k \alpha \notin \Gamma_1$

Examples.

1. $\Gamma'_1 = \Gamma'_2 = \emptyset$

2. "The shift case".

Thm (B-D classification). Let $\mathfrak{g}$ be a simple Lie alg, $(\cdot,\cdot)$ a metric, $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, then if $(\Gamma_1, \Gamma_2, \tau)$ is a B-D triple, and $r_0 \in \mathfrak{g} \otimes \mathfrak{g}$ is s.t.

1. $r^0_{12} + r^0_{21} = \Omega^0$

2. $(\tau(\alpha) \otimes 1) r^0 + (1 \otimes \alpha) r^0 = 0 \quad \forall \alpha \in \Gamma_1$

then there is a corresponding non-skew-symm. structure on $\mathfrak{g}$, and this correspondance is bijective.

⋮