Halacheva's proof of IHX for the Duzhin weight system

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The IHX relation:

$$\det(x|z|y)\det(z|u|v) - \det(x|u|z)\det(v|y|z) + \det(z|y|u)\det(v|x|z) = 0$$

This is equivalent to (after some column permutations):

$$\det(y|x|z)\det(z|u|v) + \det(u|x|z)\det(z|v|y) + \det(v|x|z)\det(z|y|u) = 0$$

Translating this in the language of dot and cross products, we have:

$$\begin{aligned}(y\cdot(x\times z))(z\cdot(u\times v)) + (u\cdot(x\times z))(z\cdot(v\times y)) + (v\cdot(x\times z))(z\cdot(y\times u)) &= 0\\ \iff ((z\cdot(u\times v))y + (z\cdot(v\times y))u + (z\cdot(y\times u))v)\cdot(x\times z) &= 0\end{aligned}$$

Using the relation $a\times(b\times c) = b(a\cdot c) - c(a\cdot b)$[1], we notice that:

$$\begin{aligned}z\times(y\times(u\times v)) &= y(z\cdot(u\times v)) - (u\times v)(z\cdot y)\\ z\times(u\times(v\times y)) &= u(z\cdot(v\times y)) - (v\times y)(z\cdot u)\\ z\times(v\times(y\times u)) &= v(z\cdot(y\times u)) - (y\times u)(z\cdot v)\end{aligned}$$

So IHX becomes:

$$(z\times(y\times(u\times v)+u\times(v\times y)+v\times(y\times u))+((u\times v)(z\cdot y)+(v\times y)(z\cdot u)+(y\times u)(z\cdot v)))\cdot(x\times z) = 0$$

We have an equation of the form $(A+B)\cdot(x\times z) = 0$ and we claim that $B\cdot(x\times z) = 0$. This can be seen using the formula $(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$[2]:

$$\begin{aligned}(z\cdot y)(u\times v)\cdot(x\times z) &= (z\cdot y)(\underline{(u\cdot x)(v\cdot z)} - \underline{(u\cdot z)(v\cdot x)})\\ (z\cdot u)(v\times y)\cdot(x\times z) &= (z\cdot u)(\underline{(v\cdot x)(y\cdot z)} - \underline{(v\cdot z)(y\cdot x)})\\ (z\cdot v)(y\times u)\cdot(x\times z) &= (z\cdot v)(\underline{(y\cdot x)(u\cdot z)} - \underline{(y\cdot z)(u\cdot x)})\end{aligned}$$

The corresponding terms cancel each other. Therefore, the IHX relation is equivalent to:

$$\begin{aligned}(z\times(y\times(u\times v)+u\times(v\times y)+v\times(y\times u)))\cdot(x\times z) &= 0\\ \iff (z\times([y,[u,v]]+[u,[v,y]]+[v,[y,u]]))\cdot(x\times z) &= 0\end{aligned}$$

[1] The vector triple product formula,Cross product.
[2] The three-dimensional case of the Binet-Cauchy identity, see Cross product.