Maria Podkopaeva on Rouviere and Symmetric Spaces

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$ch(x,y) = \log(e^x e^y) \in lie_2$

$ch(ch(x,y), z) = ch(x, ch(y,z))$

& ch is the unique "associative" series which begins w/ $x + y + \frac{1}{2}[x,y] + \dots$

Rouviere: $e^x e^y = e^{w(x,y)} h(x,y)$

$w(x,y) = -w(-x,-y) \quad h(x,y) = h(-x,y)$

$w = \frac{1}{2}\log e^x e^{2y} e^x$

"Associativity":

$w(x, w(y,z)) = w(w(x,y), h(x,y) z h^{-1}(x,y))$

$h(w(x,y), h z h^{-1}) h(x,y) = h(x, w(y,z)) h(y,z)$

<u>Claim</u> If f satisfies w's eqn's & $f = x + y + \dots$ then $f = w$ [for a fixed h].

$tder_n$: $u \in der(lie_n)$ s.t. $u(x_i) = [x_i, a_i]$

$\rightsquigarrow TAut_n$

$tr_n := Ass_n / ab = ba \simeq$ cyclic words.

$tr : Ass_n \to tr_n$ "the trace"

$str_n := Ass_n / ab \sim (-1)^{|a||b|} ba$

$str : Ass_n \to str_n$ "the super trace"

$\alpha : tder_n \to tr_n$ by

$(a_1, \dots, a_n) \mapsto tr(\sum a_i)$

$s\alpha : tder_n \to str_n$ by

$(a_1, \dots, a_n) \mapsto str(\sum a_i)$

$div: tder_n \to tr_n$ by

$(a_1 \dots a_n) \mapsto tr(\sum x_i \partial_{x_i} a_i)$

where if $a = a_0 + \sum a_i x_i$ then

$\partial_{x_i} a := a_i$

$sdiv: (a_1 \dots a_n) \mapsto str(\sum x_i \partial_{x_i} a_i)$

<u>Prop</u> All 4 are 1-cocycles.

They integrate to A, sA, j, sj.

$(\delta F)(x,y) = F(y) - F(ch(x,y)) + F(x)$

$(\delta^s F)(x,y) = F(y) - F(w(x,y)) + F(x)$

KV: $\exists F \in TAut_2$ s.t.

1. $F(x+y) = ch(x+y)$
2. $e(x,y) := j(F) \in im(\delta)$

Rouviere: $\exists F \in TAut_2^{even}$ s.t.

1. $F(x+y) = w(x,y)$
2. $e(x,y) \in im\, \delta^s$ w/ $e = j(F)$

<u>Thm1(AET)</u> $\Phi \in Assoc_1$

$F \in TAut_2$ s.t. $F^{-1} = \mu: \begin{array}{l} x \mapsto \lambda_1 x \lambda_1^{-1} \\ y \mapsto \lambda_2 y \lambda_2^{-1} \end{array}$

where $\lambda_1 = \Phi(x, -x-y)$

$\lambda_2 = e^{-\frac{x+y}{2}} \Phi(y, -x-y)$

Then $F^{1,2} F^{12,3} = F^{2,3} F^{1,23} \Phi^{1,2,3}$

Thm2 & F solves KV

Thm2 & F solves KV.

Thm 1' Given $\Phi \in \mathrm{Assoc}_2^{\mathrm{even}}$ (the subscript 2 means $\mu = 2$ in $K = e^{\mu/2 t}$)

set $F \in \mathrm{TAut}_2$ s.t.

$$F^{-1} = \mu : \begin{array}{l} x \mapsto \lambda_1 x \lambda_1^{-1} \\ y \mapsto \lambda_2 y \lambda_2^{-1} \end{array}$$

$\lambda_1 = \Phi(x, -x-y)$

$\lambda_2 = \Phi(x, -x-y)\Phi(y, x)$ then

$$H^{1,2,3} F^{1,2} F^{12,3} = F^{2,3} F^{1,23} \Phi^{1,2,3}$$

$$H : \begin{array}{l} x \mapsto x \\ y \mapsto y \\ z \mapsto h z h^{-1} \end{array}$$

Thm 2' Such F solves Rouviere.

Proof of 1' Using Thm1

$$\nu : \begin{array}{l} x \mapsto x \\ y \mapsto e^{x} y e^{-x} \end{array} \qquad \nu(w) = ch(x, y)$$

$$\begin{tikzcd} w(x,y) \arrow[r, mapsto, "\nu"] \arrow[rr, bend right, "\tilde{\mu}"'] & ch \arrow[r, mapsto, "\mu"] & x+y \end{tikzcd}$$

Hexagon $\Rightarrow$ $\tilde{\mu}$ satisfies Thm 1'.

$$\nu^{1,2} \nu^{12,3} = \nu^{2,3} \nu^{1,23} H^{1,2,3}$$

follows from assoc. of ch, w

$\Rightarrow \tilde{\mu}\tilde{\mu} H = \Phi \tilde{\mu}\tilde{\mu}$