Handout as of May 7

May-07-11
6:45 PM

Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

Dror Bar−Natan at Swiss Knots 2011

http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

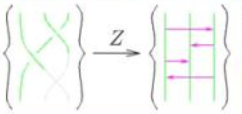
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle\ker\mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.


Peter Lee

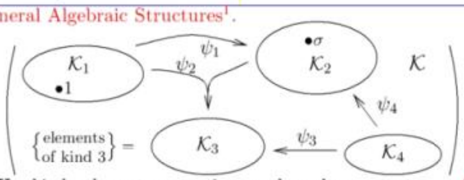
The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way.

Example 2. For $K = \mathbb{Q}PvB_n =$ "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works!

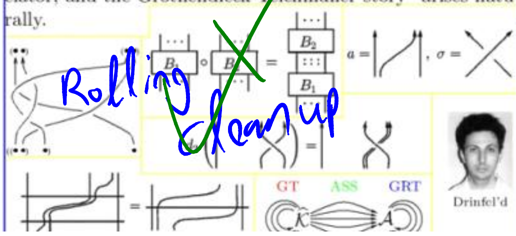
Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

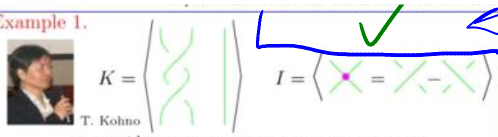
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an associator, and the Grothendieck-Teichmuller story[2] arises naturally.



Rolling cleanup

Example 1.



$(K/I^{m+1})^\star =$ (invariants of type m) $=: \mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1}$ $\quad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle = \langle \ \rangle$

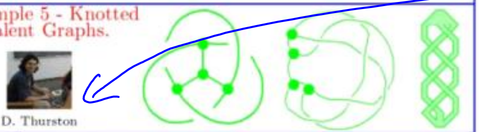
$\ker\mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle$4T relations$\rangle$

$\mathcal{A} = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \left\langle \ \right\rangle / 4T$

Z: universal finite type invariant, the Kontsevich integral.

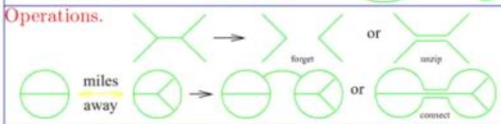
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

Example 5 - Knotted Trivalent Graphs.

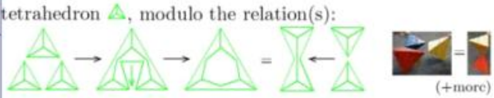


Add ref to his paper

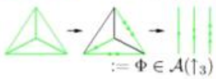
Operations.



Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):



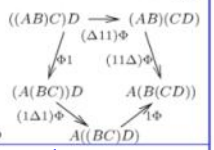
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.



An Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",



$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

Diagrams to $\mathcal{U}(\mathfrak{g})$ goes here

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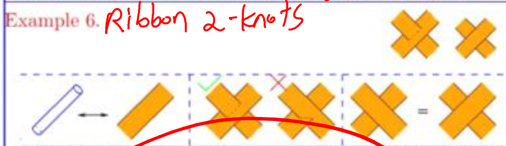
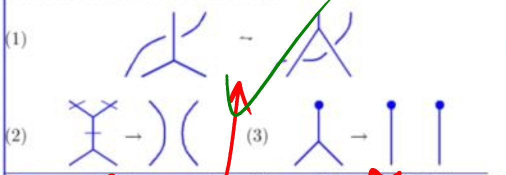
Example 6. Ribbon 2-knots

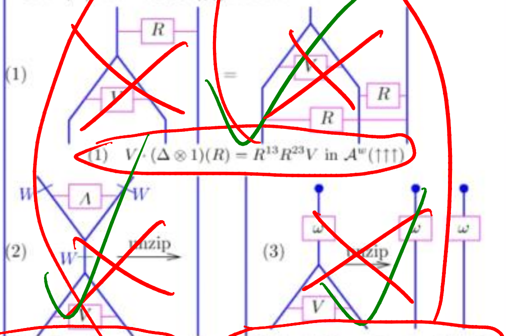


The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp\mathcal{H} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V \cdot (\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$

(3) $V \cdot \Delta(\omega) = \omega \otimes \omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F \in \mathrm{TAut}_2$ and $a \in \mathrm{tr}_1$ such that

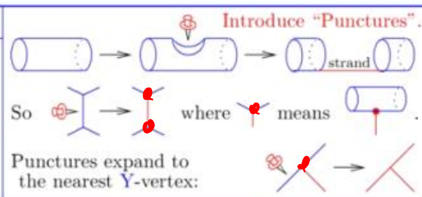
$$F(x+y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Add AT-KV Pictures

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Footnotes and references are on the PDF version, page 3.

Introduce "Punctures".



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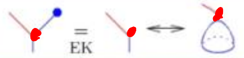
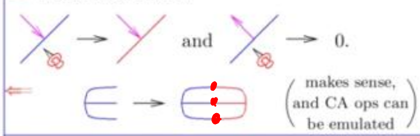
Note.



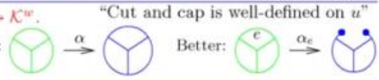
$\mathcal{K}^w$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:

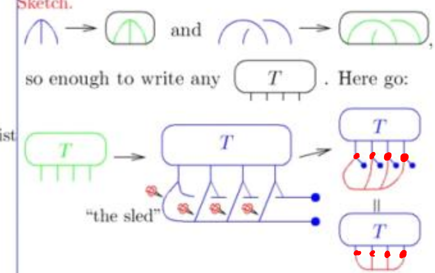


$\mathcal{K}^u \to \mathcal{K}^w$. "Cut and cap is well-defined on u"



Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch.



Rachmi white space

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention $P vB$, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minute) Example: parenthesized braids and horizontal associators.
6. (6 minute) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (5 minute) Example: wKO's and the Kashiwara-Vergne equations.
8. (15 minute) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minute) wKO's, uKO's, and Alekseev-Enriquez-Torrosian.
10. (1 minute) The third page.

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. See my paper [BN1] and my talk/handout/video [BN2].

References

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the Yang-Baxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Braids and the Grothendieck-Teichmuller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.