Handout as of May 20

May-20-11
10:09 AM

* Partition! * Mark minutes (in pink)!

Facts and Dreams About v-Knots and Etingof-Kazhdan, 1

Dror Bar-Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/
Foots & refs on PDF version, page 3.

This is an overview with too many and not enough details. I apologize.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

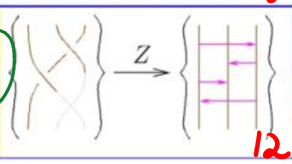
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\mathrm{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11}\rangle$ of K surjects using μ on $\mathrm{gr}\, K$.

Peter Lee

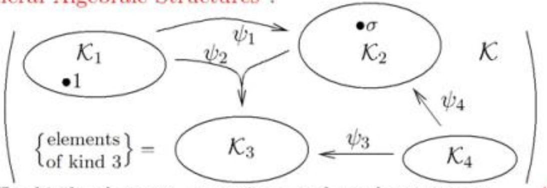
The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z\colon K \to \mathcal{A}$ for which $\mathrm{gr}\, Z : \mathrm{gr}\, K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one. 12



General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works! 14

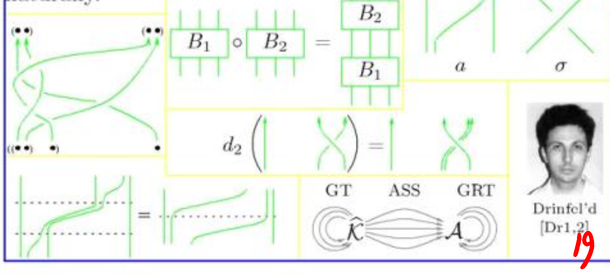
Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \quad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

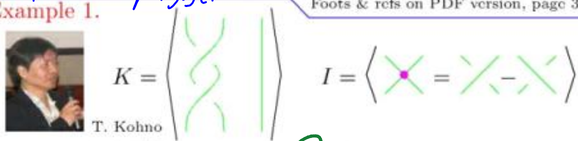
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$ 15

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[2] arises naturally.



Drinfel'd [Dr1,2] 19

Example 1.



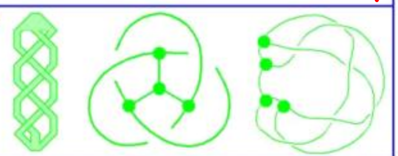
T. Kohno

$$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$$
$$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1}, \quad C = \langle t^{ij} \mid t^{ij} = t^{ji}\rangle$$
$$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle 4T \text{ relations}\rangle$$
$$\mathcal{A} = A_n = \left(\begin{matrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{matrix}\right)$$

Z: universal finite type invariant, the Kontsevich integral. 9

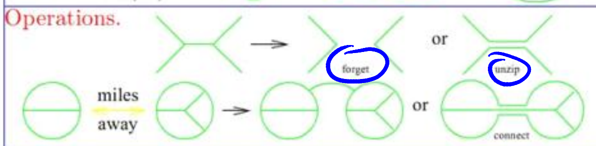
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

Example 5 - Knotted Trivalent Graphs.



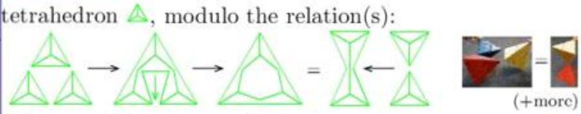
D. Thurston [Th]

Operations.



raise by ε.

Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):



(+more)

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras. $:= \Phi \in \mathcal{A}(\uparrow_3)$

A $\mathcal{U}(\mathfrak{g})$-Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

$\mathcal{A}(\uparrow_2) :=$ (diagrams) / AS, IHX, STU; $(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$

Given a metrized $\mathfrak{g}$, $\mathfrak{g} = \langle X_a \rangle$ $\to \mathcal{U}(\mathfrak{g})^{\otimes 2}$

$$\to \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

Penrose, Cvitanovic 25

Facts and Dreams About v-Knots and Etingof-Kazhdan, 2

Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".

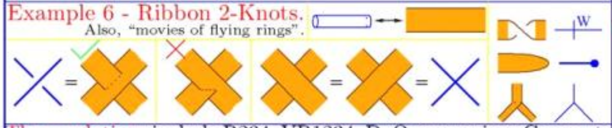
Forbidden Theorem [EK, Ha, ?]. There exists a homomorphic expansion Z for vTT.

Why Forbidden?

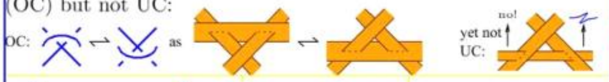
- Minor statement details may be off.

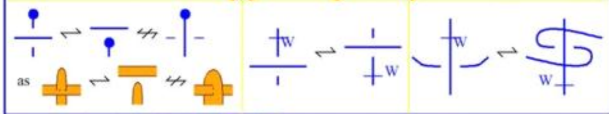
Facts and Dreams About v–Knots and Etingof–Kazhdan, 2

Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



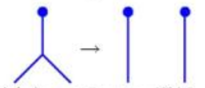


Trivalent w-Tangles. 30

wTT = PA ⟨ w-generators | w-relations | unary w-operations ⟩ = CA ⟨ same w/o × ⟩

Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:

$V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow_3)$ $\qquad VV^* = I$ in $\mathcal{A}^w(\uparrow_2)$



$V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow_2)$

Kashiwara-Vergne-Alekseev-Enriquez-Torrossian

Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F\in \mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

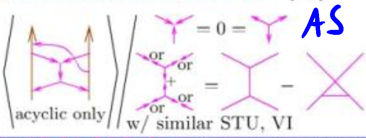
$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y).$$ 33

Theorem. That's equivalent to a homomorphic expansion for wTT.

The Main Example.

vTT = PA ⟨  | R234, VR234, D, yet not UC, OC | unzips ⟩ = CA ⟨ same w/o × ⟩

The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po]. AS

$\mathcal{A}^v \simeq$ 

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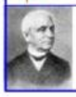
$\mathcal{A}^v$ pairs with Lie bialgebras. Let $\mathfrak{g}_+$ be a Lie bialgebra with basis X_a, bracket [], cobracket δ, dual $\mathfrak{g}_- = \mathfrak{g}_+^*$, dual basis X^a for $\mathfrak{g}_-$, double $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-$, structure constants $[X_a,X_b]=\sum b_{ab}^c X_c$ and co-structure constants $\delta(X_a)=\sum c_a^{bc}X_b\otimes X_c$. Then

$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} b_{de}^c c_c^{ba} X_a X^d X_f \otimes X_b X^f X^e \in \mathcal{U}(\mathfrak{g})^{\otimes 2}$$ 39



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org 

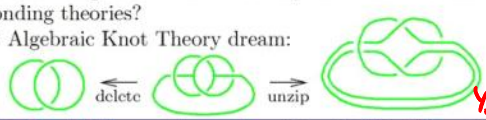
Forbidden Theorem [EK, Ha, ?]. There exists a homomorphic expansion Z for vTT.

Why Forbidden?

- Minor statement details may be off.
- No fully written proof.
- I don't understand the proof.
- There isn't yet a knot-theoretic view of the proof, like there is in the w-case. 42

Why Should We Care?

- A gateway into the forbidden territory of "quantum groups".
- Abstractly more pleasing: We study the things, and not just their representations.
- $\mathcal{A}^v$ is sometimes easier than $\mathcal{A}^u$: Alexander, say, arises easily from the 2D Lie algebra[3].
- Potentially, $\mathcal{A}^v$ has many more "internal quotients" than there are Lie bialgebras. What are they and what are the corresponding theories?
- My old[4] Algebraic Knot Theory dream:



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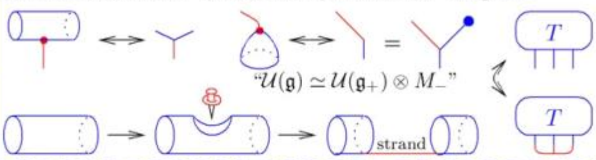
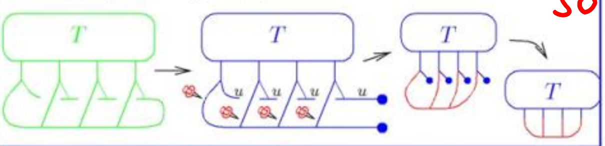
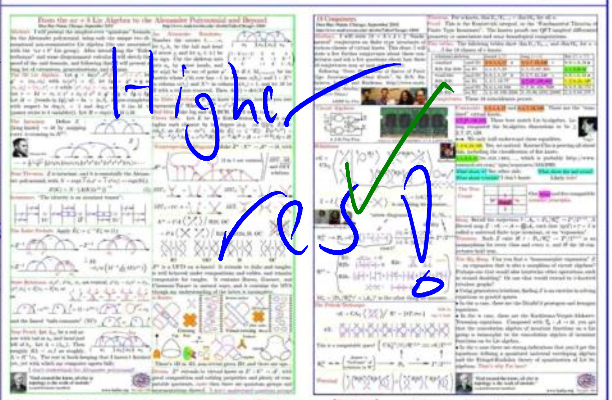
$V\to\Phi^{\text{1-loop}}$ after [AT]. "cut and cap" is well-defined(!) on $\mathcal{K}^u$

$\Phi\to V$ after [AET]. In $\mathcal{K}^{\overline{w}}$ allow tubes and strands and tube-strand vertices, allow "punctures", yet allow no "tangles".



The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V). With T any classical tangle, esp. (T) or (T), consider the "sled" 50





Highly rec'd

Alexander is easy! In Chicago, [BN4] Many kinds of virtuals!

Help Needed!

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. See my paper [BN1] and my talk/handout/video [BN3].
3. See [BN5] and my talk/handout/video [BN4].
4. Not so old and not quite written up. Yet see [BN2].

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, Braid groups and explicit solutions of the Kashi-wara Vergne equations*, Pub. Math. de L'IHES **112-1** (2010) 143–189, arXiv:arXiv:0903.4067.

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[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmüller Group I*, Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Algebraic Knot Theory — A Call for Action*, web document, 2006, `http://www.math.toronto.edu/~drorbn/papers/AKT-CFA.html`.

[BN3] D. Bar-Natan, *Braids and the Grothendieck-Teichmüller Group*, talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[BN4] D. Bar-Natan, *From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond*, talk given in Chicago on September 11, 2010, `http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/`.

[BN5] D. Bar-Natan, *Finite Type Invariants of w-Knotted Objects: From Alexander to Kashiwara and Vergne*, in preparation, online at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

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[Th] D. P. Thurston, *The Algebra of Knotted Trivalent Graphs and Turaev's Shadow World*, Geometry & Topology Monographs **4** (2002) 337-362, arXiv:math.GT/0311458.

Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention $P\!vB$, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minute) Example: parenthesized braids and horizontal associators.
6. (6 minute) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (5 minute) Example: wKO's and the Kashiwara-Vergne equations.
8. (15 minute) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minute) wKO's, uKO's, and Alekseev-Enriquez-Torrosian.
10. ~~(1 minute) The third page.~~