Handout as of May 14

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Facts and Dreams About v−Knots and Etingof−Kazhdan, 1

Dror Bar–Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/
Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\mathrm{gr}_I K := \bigoplus I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11}\rangle$ of K surjects using μ on $\mathrm{gr}\, K$.

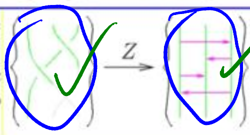

Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z\colon K \to \mathcal{A}$ for which $\mathrm{gr}\, Z\colon \mathrm{gr}\, K \to \mathcal{A}$ inverts μ.

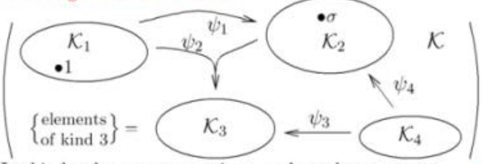
Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way.

decide on a v-colour and turn PvB into it.

Example 2. For $K = \mathbb{Q}PvB_n =$ "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works!

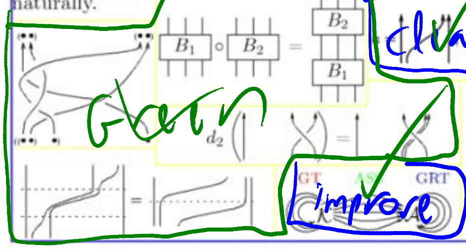
Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1\wedge x = 1,\quad x\wedge 1 = x = x\wedge x, \qquad \text{(appetizers)}$$
$$(x\wedge y)\wedge z = (x\wedge z)\wedge(y\wedge z). \qquad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x}\wedge\bar{y})\wedge\bar{z} = (\bar{x}\wedge\bar{z})\wedge\bar{y} + \bar{x}\wedge(\bar{y}\wedge\bar{z}).$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n-associator, and the Grothendieck-Teichmüller story[2] arises naturally.




Drinfel'd [Dr1,2]

Example 1.


T. Kohno

$K = \langle \text{pure braids} \rangle \qquad I = \langle \text{crossing} - \text{crossing} \rangle$

$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \qquad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle$

$\ker\mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle 4\mathrm{T}\ \text{relations}\rangle$

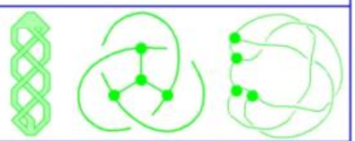
$\mathcal{A} = A_n = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \langle \ \rangle / 4\mathrm{T}$

Z: universal finite type invariant, the Kontsevich integral.

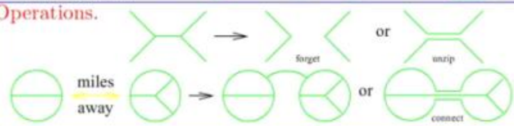
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

Example 5 - Knotted Trivalent Graphs.

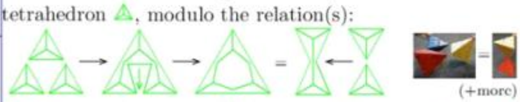

D. Thurston [Th]



Operations.



Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):

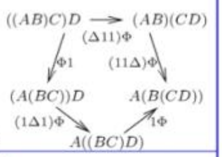


Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

A $\mathcal{U}(\mathfrak{g})$-Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",



$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

$\mathcal{A}(\uparrow_2) := \langle \ \rangle / \mathrm{AS}, \ \mathrm{IHX}$

$(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$

Given a metrized $\mathfrak{g}$, $\mathfrak{g} = \langle X_a\rangle$ $\longrightarrow \mathcal{U}(\mathfrak{g})^{\otimes 2}$


Penrose


Cvitanovic

$$\longrightarrow \sum_{a,b,c,d,e,f=1}^{\dim\mathfrak{g}} f_{abc}f_{dce}X_aX_dX_f\otimes X_bX_fX_e$$

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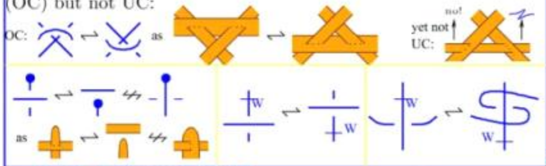
Example 6 - Ribbon 2-Knots. Also, "movies of flying rings".

The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

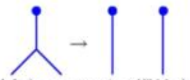


Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \begin{matrix}\text{w-}\\ \text{generators}\end{matrix} \middle| \begin{matrix}\text{w-}\\ \text{relations}\end{matrix} \middle| \begin{matrix}\text{unary w-}\\ \text{operations}\end{matrix} \right\rangle$$

Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:

$V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow_3)$ $\qquad$ $VV^* = I$ in $\mathcal{A}^w(\uparrow_2)$



$V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow_2)$

Kashiwara-Vergne-Alekseev-Enriquez-Torrosian

Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F\in \text{TAut}_2$ and $a\in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x)+a(y)-a(\log e^x e^y).$$

Theorem. That's equivalent to a homomorphic expansion for wTT.

The Main Course.

$$\text{vTT}^a = \widetilde{\text{CA}}\left\langle \begin{matrix}\text{v-}\\ \text{generators}\end{matrix} \middle| \begin{matrix}\text{v-}\\ \text{relations}\end{matrix} \middle| \begin{matrix}\text{unary v-}\\ \text{operations}\end{matrix} \right\rangle$$

Forbidden Theorem. There exists a homomorphic expansion Z for vTT^a.

Why Forbidden?
- Minor statement details may be off.
- I don't understand the proof.
- There isn't yet a knot-theoretic view of the proof, like there is in the w-case

Etingof

Kazhdan

The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po].

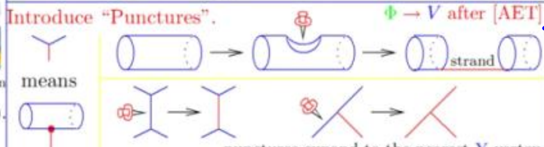
Polyak $\quad$ Ohtsuki

write & swap w/ previous block.

~~Challenges and Hopes.~~ Why should you and I care?
- A gateway into the forbidden territory of "Quantum Groups".
- Abstractly more pleasing: we study the things and not just their representations.
- Potentially, A^v has many more quotients/images then there are quantum groups. What are they?
- My old Alg. Knot Th. dream $\boxed{\text{picture here}}$

$V \to \Phi^{1\text{-loop}}$ after [AT]. "cut and cap" is well-defined(!) on $\mathcal{K}^u$

Introduce "Punctures". $\Phi\to V$ after [AET]

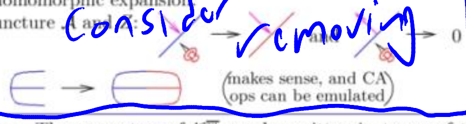


Note.

$\mathcal{K}^{\bar w}$. Allow tubes and strands and tube-strand vertices, yet allow only "compact" knots — no runs to ∞.

$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\bar w}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.
$\Rightarrow$ Puncture A and Z:

consider removing

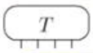


$\Leftarrow$ (makes sense, and CA ops can be emulated)

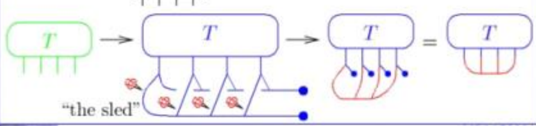
Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V).

Sketch.

and, so enough to write any



. Here go:



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. See my paper [BN1] and my talk/handout/video [BN2].

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, Braid groups and explicit solutions of the Kashi-waraVergne equations*, Pub. Math. de L'IHES **112-1** (2010) 143–189, arXiv:arXiv:0903.4067.

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[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmüller Group I*, Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Braids and the Grothendieck-Teichmüller Group*, talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[Dr1,2] V. G. Drinfel'd, *Quasi-Hopf Algebras*, Leningrad Math. J. **1** (1990) 1419–1457 and *On Quasitriangular Quasi-Hopf Algebras and a Group Closely Connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. **2** (1991) 829–860.

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[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions*, Invent. Math. **47** (1978) 249–272.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic*, in preparation.

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[Th] D. P. Thurston, *The Algebra of Knotted Trivalent Graphs and Turaev's Shadow World*, Geometry & Topology Monographs **4** (2002) 337-362, arXiv:math.GT/0311458.

Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention PcB, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minute) Example: parenthesized braids and horizontal associators.
6. (6 minute) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (5 minute) Example: wKO's and the Kashiwara-Vergne equations.
8. (15 minute) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minute) wKO's, uKO's, and Alekseev-Enriquez-Torrosian.
10. (1 minute) The third page.