Handout as of May 12

May-12-11

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Facts and Dreams About v–Knots and Etingof–Kazhdan, 1

Dror Bar–Natan at Swiss Knots 2011

http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

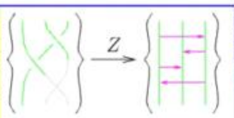
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\operatorname{gr}_I K := \bigoplus I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11}\rangle$ of K surjects using μ on $\operatorname{gr} K$.

Peter Lee

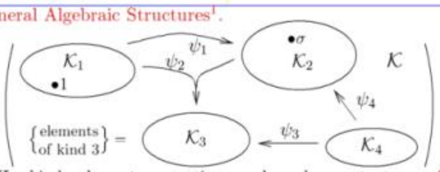
The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z\colon K \to \mathcal{A}$ for which $\operatorname{gr} Z : \operatorname{gr} K \to \mathcal{A}$ inverts μ.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way.

Example 2. For $K = \mathbb{Q}PuB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works!

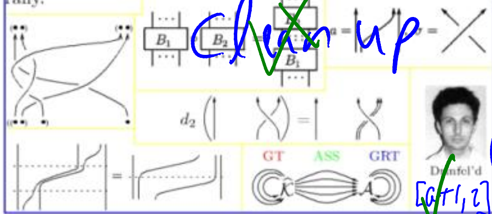
Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \quad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad \text{(main)}$$

$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an associator, and the Grothendieck-Teichmuller story[2] arises naturally.

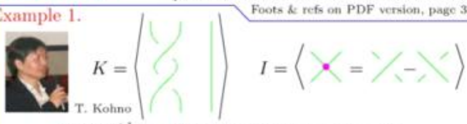


clean up

[GT1,2]

Drinfel'd

Example 1.



$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji}\rangle$

$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}]\rangle = \langle 4\text{T relations}\rangle$

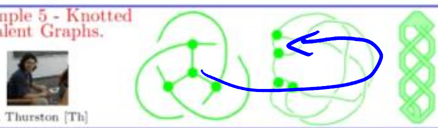
An

$\mathcal{A} = \left(\begin{matrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{matrix}\right) = \Big/ 4\text{T}$

Z: universal finite type invariant, the Kontsevich integral.

Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

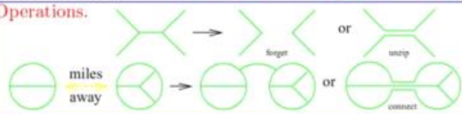
Example 5 - Knotted Trivalent Graphs.



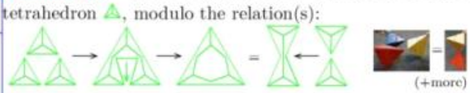
D. Thurston [Th]

cycle images

Operations.



Presentation. KTG is generated by ribbon twists and the tetrahedron, modulo the relation(s):



Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

$=: \Phi \in \mathcal{A}(\uparrow_3)$

An Associator: why?

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{tikzcd} ((AB)C)D \arrow[r, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"] & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\ (A(BC))D \arrow[d, "(1\Delta 1)\Phi"] & A(B(CD)) \\ A((BC)D) \arrow[ur, "1\Phi"] & \end{tikzcd}$$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

$\mathcal{A}(\uparrow_2) := \Big/ \text{AS},$

Given a metrized $\mathfrak{g}$, $\mathfrak{g} = \langle X_a\rangle$ $\to \mathcal{U}(\mathfrak{g})^{\otimes 2}$

$(\deg = \frac{1}{2}\#\{\text{trivalent vertices}\})$

$$\mapsto \sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abc} f_{dce} X_a X_d X_f \otimes X_b X_f X_e$$

lower

Penrose

Cvitanović

bigger if possible

Also, "Flying rings"

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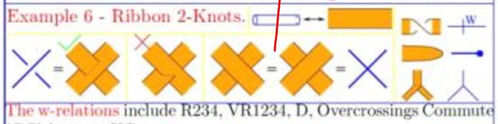
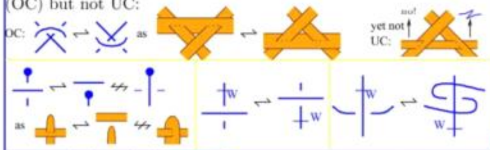
Example 6 - Ribbon 2-Knots.



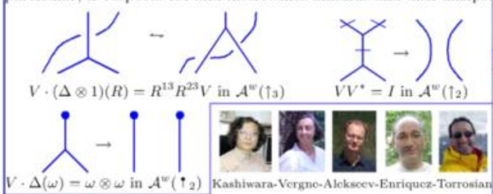
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array}\middle|\begin{array}{c}\text{w-}\\ \text{relations}\end{array}\middle|\begin{array}{c}\text{unary w-}\\ \text{operations}\end{array}\right\rangle$$

Theorem. There exists a homomorphic expansion Z for wTT. In particular, Z respects $R4$ and intertwines annulus and disk unzips:



Alekseev-Torossian [AT] (equivalent to Kashiwara-Vergne [KV]). There are elements $F \in \text{TAut}_2$ and $a \in \text{tr}_1$ such that

$F(x+y) = \log e^x e^y$ and $jF = a(x) + a(y) - a(\log e^x e^y)$.

Theorem. That's equivalent to a homomorphic expansion for wTT.

The Main Course.

$$\text{vTT}^a = \widetilde{\text{CA}}\left\langle \begin{array}{c}\text{v-}\\ \text{generators}\end{array}\middle|\begin{array}{c}\text{v-}\\ \text{relations}\end{array}\middle|\begin{array}{c}\text{unary v-}\\ \text{operations}\end{array}\right\rangle$$

Forbidden Theorem. There exists a homomorphic expansion Z for vTT^a.

$\boxed{\mathcal{E}}$

$\boxed{K}$

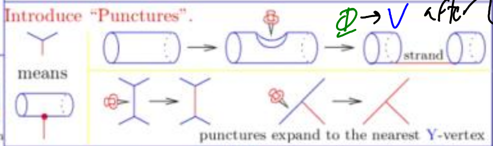
The Polyak-Ohtsuki Description of $\mathcal{A}^v$ [Po].

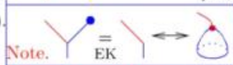
Polyak Ohtsuki

Challenges and Hopes.

Introduce "Punctures".

$\Phi \to V$ after [AET]

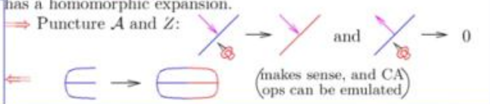




$\mathcal{K}^{\bar{w}}$. Allow tubes and strands and tube-strand vertices, yet allow only "compact" knots — no runs to ∞.

$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\bar{w}}$ equivalence. $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\bar{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:



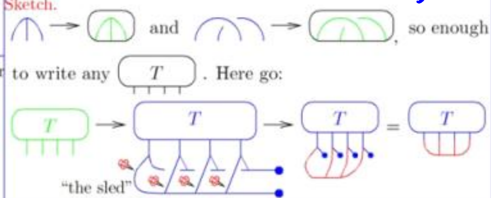
$\Longleftarrow$

$\mathcal{K}^u \to \mathcal{K}^{\bar{w}}$.

Also rearrange & put "$V \to \Phi$ after [AT]".

Theorem. The generators of $\mathcal{K}^{\bar{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V).

Sketch.



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".

2. See my paper [BN1] and my talk/handout/video [BN2].

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the Yang-Baxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Braids and the Grothendieck-Teichmuller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[EK] P. Etingof and D. Kazhdan, *Quantization of Lie Bialgebras, I,* Selecta Mathematica, New Series **2** (1996) 1–41, arXiv:q-alg/9506005.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.

[Po] M. Polyak, *On the Algebra of Arrow Diagrams,* Let. Math. Phys. **51** (2000) 275–291.

[Th] D. P. Thurston, *The Algebra of Knotted Trivalent Graphs and Turaev's Shadow World,* Geometry & Topology Monographs **4** (2002) 337-362, arXiv:math.GT/0311458.

Plan

1. (8 minutes) The Peter Lee setup for (K, I), "all interesting graded equations arise in this way".
2. (3 minutes) Example: the pure braid group (mention PvB, too).
3. (3 minutes) Generalized algebraic structures.
4. (1 minute) Example: quandles.
5. (4 minute) Example: parenthesized braids and horizontal associators.
6. (6 minute) Example: KTGs and non-horizontal associators. ("Bracket rise" arises here).
7. (5 minute) Example: wKO's and the Kashiwara-Vergne equations.
8. (15 minute) vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
9. (5 minute) wKO's, uKO's, and Alekseev-Enriquez-Torrosian.
10. (1 minute) The third page.