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Facts and Dreams About v–Knots and Etingof–Kazhdan, 1

Dror Bar–Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\mathrm{gr}_I K := \bigoplus I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}: (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11} \rangle$ of K surjects using μ on $\mathrm{gr}\, K$.

Peter Lee

The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z: \hat{K} \to \mathcal{A}$ inducing the identity on $I/I^2 = C$.

Dror's Dream. All interesting objects and equations, especially those around quantum groups, arise this way.

definitions mismatch.

Example 1. $K = \langle \; \rangle$ $I = \langle \times = \; - \; \rangle$

T. Kohno

$(K/I^{m+1})^\star = (\text{invariants of type } m) =: \mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1}$ $C = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \; \rangle$

$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4T \text{ relations} \rangle$

$\mathcal{A} = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \langle \; \rangle / 4T$

Z: universal finite type invariant, the Kontsevich integral.

Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look" [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.

Example 5. KTGs

D. Thurston

light green interior

a bit denser

operations

forget, or, unzip

miles away, or, connect

$(A\otimes B)\otimes C \xrightarrow[\text{in } U(\mathfrak{g})^{\otimes 3}]{\Phi} A\otimes(B\otimes C)$

$((A\otimes B)\otimes C)\otimes D \xrightarrow{(\Delta\otimes 1\otimes 1)\Phi} (A\otimes B)\otimes(C\otimes D) \xrightarrow{(1\otimes 1\otimes\Delta)\Phi} A\otimes(B\otimes(C\otimes D))$

$\Phi\otimes 1$, $(A\otimes(B\otimes C))\otimes D$, $(1\Delta 1)\Phi$, $A\otimes((B\otimes C)\otimes D)$, $1\otimes\Phi$, $A\otimes(B\otimes(C\otimes D))$

Φ is an "associator"

$(\Phi\otimes 1)\cdot(1\Delta 1)\Phi\cdot(1\otimes\Phi) = (\Delta 1 1)\Phi\cdot(1 1\Delta)\Phi$

"The Drinfel'd Pentagon"

TeX and move

Using moves, KTG is generated by ribbon twists and the tetrahedron: in TeX

Modulo the relation(s): TeX

(+more) others

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

$\Leftrightarrow \Phi \in \mathcal{A}(\uparrow_3)$

smaller font.

Add a "diagrams to $U(\mathfrak{g})$" box.

General Algebraic Structures[1].

$\mathcal{K}_1$ •1, ψ_2, ψ_1, •σ $\mathcal{K}_2$, $\mathcal{K}$, ψ_4, $\left\{\begin{matrix}\text{elements}\\ \text{of kind } 3\end{matrix}\right\} = \mathcal{K}_3$, ψ_3, $\mathcal{K}_4$

- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works!

Example 3. Quandle: a set K with an op $\wedge$ s.t.

$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x,$ (appetizers)

$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z).$ (main)

$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

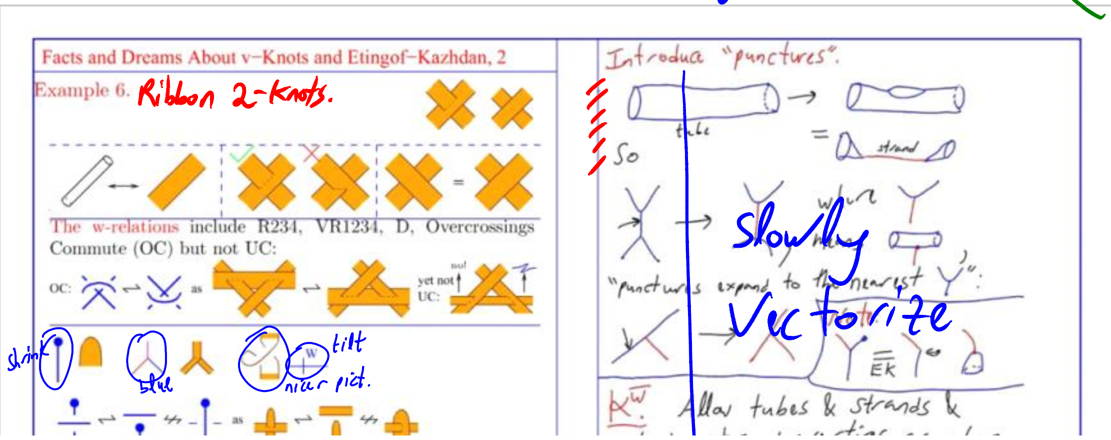
$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an associator, and the Grothendieck-Teichmuller story[2] arises naturally.

Rolling upgrade

B_1, B_2, $a =$, $\sigma =$, d_2, GT, ASS, GRT, K, $\mathcal{A}$

Drinfel'd



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Example 6. Ribbon 2-Knots.

The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

OC: as — yet not UC:

shrink, blue, tilt, nicer pict.

Introduce "punctures".

tube, = strand

So

when "punctures expand to the nearest Y":

Slowly Vectorize

K^w: Allow tubes & strands &

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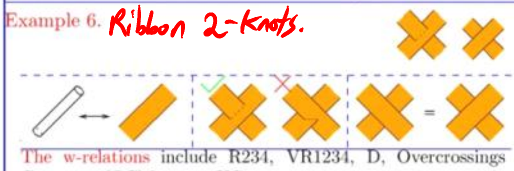
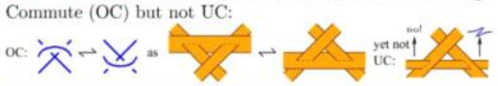
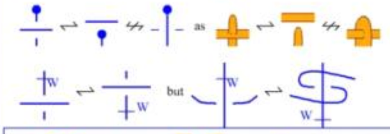
Example 6. Ribbon 2-Knots.

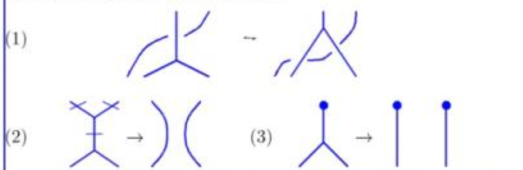


The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:

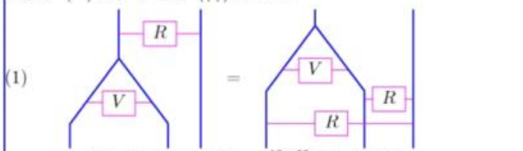




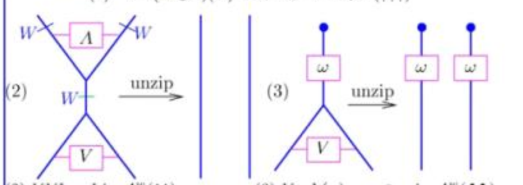
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \leftrightarrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V \cdot (\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$



(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V \cdot \Delta(\omega) = \omega \otimes \omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

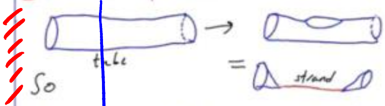
Alekseev-Torossian statement. There are elements $F \in \mathrm{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

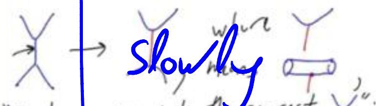
$$F(x+y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Footnotes and references are on the PDF version, page 3.

Introduce "punctures".





"punctures expand to the nearest Y":

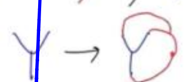
Slowly
Vectorize

$K^{\bar{w}}$: Allow tubes & strands & tube-strand vertices as above, but allow only compact knots — nothing runs to ∞.

$K^{\bar{w}} \leftrightarrow K^w$: Claim K^w has a homomorphic expansion iff $K^{\bar{w}}$ has a homomorphic expansion.

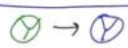
⟹ Puncture A & Z:

⟸

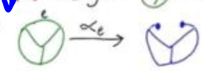


(makes sense, and CA ops can be emulated.)

$K^u \to K^{\bar{w}}$: Light:



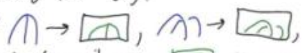
Better:



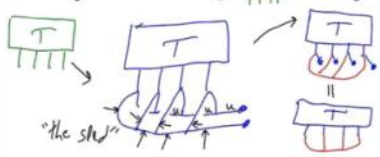
"Cut & cap is well-defined on u"

Theorem. The generators of $K^{\bar{w}}$ can be written in terms of the generators of K^u (I.e., given Φ, can write a formula for V).

Sketch:



so enough to write any T. Here go:



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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To Do

- Example: KTGs and non-horizontal associators. ("bracket rise" arises here).
- Example: wKO's and the Kashiwara-Vergne equations.
- vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
- wKO's, uKO's, and Alekseev-Enriquez-Torrosian.

Put this here.

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. See my paper [BN1] and my talk/handout/video [BN2].

References

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the YangBaxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Braids and the Grothendieck-Teichmuller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.