Handout as of April 30

April-30-11
7:02 PM

Facts and Dreams About v–Knots and Etingof–Kazhdan, 1

Dror Bar–Natan at Swiss Knots 2011
http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

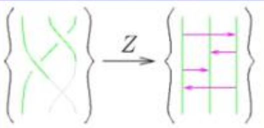
Abstract Generalities. (K, I): an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the "I-adic completion". $\text{gr}_I K := \bigoplus I^m/I^{m+1}$ has a product μ, especially, $\mu_{11}\colon (C = I/I^2)^{\otimes 2} \to I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \widehat{FC}/\langle \ker \mu_{11} \rangle$ of K surjects using μ on gr K.


Peter Lee

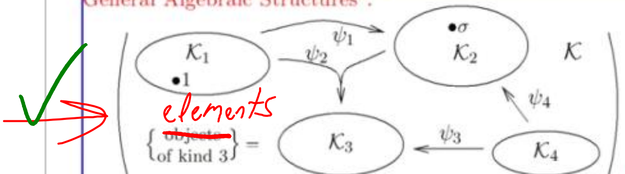
The Prized Object. A "homomorphic $\mathcal{A}$-expansion": a homomorphic filtered $Z\colon K \to \mathcal{A}$ inducing the identity on $I/I^2 = C$.

Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way.

Example 2. For $K = \mathbb{Q}PvB_n$ = "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



General Algebraic Structures[1].



- Has kinds, elements, operations, and maybe constants.
- Must have "the free structure over some generators".
- We always allow formal linear combinations.

All still works!

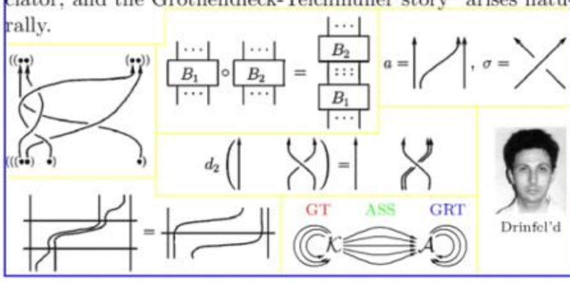
Example 3. Quandle: a set K with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

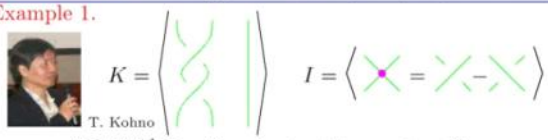
$\mathcal{A}(K)$ is a graded Lie algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I!), feed $1+\bar{x}$, $1+\bar{y}$, $1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an associator, and the Grothendieck-Teichmuller story[2] arises naturally.



Example 1.


T. Kohno

$K = \langle \ldots \rangle \quad I = \langle \ldots \rangle$

$(K/I^{m+1})^\star$ = (invariants of type m) =: $\mathcal{V}_m$

$(I^m/I^{m+1})^\star = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \ldots \rangle$

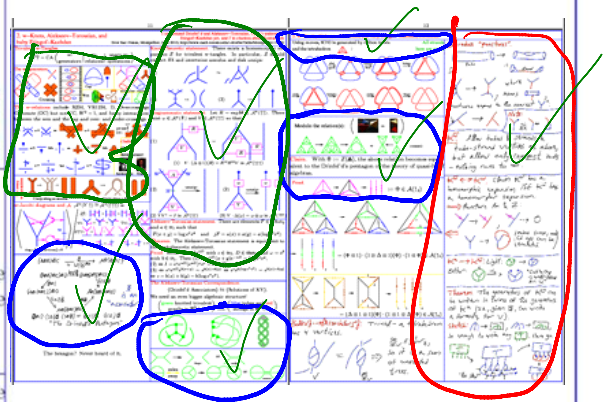
$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle \text{4T relations} \rangle$

$\mathcal{A} = \begin{pmatrix}\text{horizontal chord dia-}\\ \text{grams mod 4T}\end{pmatrix} = \langle \ldots \rangle / 4T$

Z: universal finite type invariant, the Kontsevich integral.

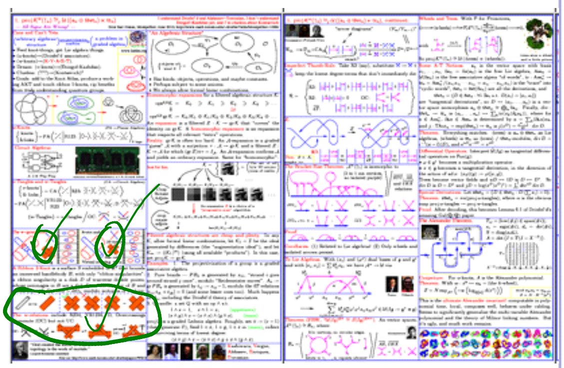
Why Prized? Sizes K and shows it "as big" as $\mathcal{A}$; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients.

Bring KTG's from Montpellier.



M2

Facts and Dreams About v–Knots and Etingof–Kazhdan, 2



ml

W-1 ✓

W-2 ✓

To Do.

- Example: KTGs and non-horizontal associators. ("bracket rise" arises here).
- Example: wKO's and the Kashiwara-Vergne equations.
- vKO's, bi-algebras, E-K, what would it mean to find an expansion, why I care (stronger invariant, more interesting quotients).
- wKO's, uKO's, and Alekseev-Enriquez-Torrosian.

Footnotes and references are on the PDF version, page 3.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Footnotes

1. I probably mean "a functor from some fixed "structure multi-category" to the multi-category of sets, extended to formal linear combinations".
2. See my paper [BN1] and my talk/handout/video [BN2].

References

[BEER] L. Bartholdi, B. Enriquez, P. Etingof, and E. Rains, *Groups and Lie algebras corresponding to the YangBaxter equations,* Jornal of Algebra **305-2** (2006) 742-764, arXiv:math.RA/0509661.

[BN1] D. Bar-Natan, *On Associators and the Grothendieck-Teichmuller Group I,* Selecta Mathematica, New Series **4** (1998) 183–212.

[BN2] D. Bar-Natan, *Braids and the Grothendieck-Teichmuller Group,* talk given in Toronto on January 10, 2011, `http://www.math.toronto.edu/~drorbn/Talks/Toronto-110110/`.

[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation.