A Strange Weight System by Duzhin - Response

May-31-11
3:35 AM

Sergei,

I am traveling now and hence math happens at slow motion. Sorry.

My recollections from Calgary in 2001 are not very clear. I remember you showed some weight system in your talk, I remember it had a "simpler version" which you could show was equivalent to the full version modulo some continuity assumptions, and I remember showing that the simpler version (not the full version) was equivalent to sl(2). But that was long enough ago that I remember no details and I had to start from scratch.

I have nothing to say about the current "full version", with a general F. It may or may not be the same of the "full version" of 2001, and already back then I had nothing to say (though you said full was equivalent to simpler modulo some continuity for F, I never knew why). So everything I will say below is about the simpler version, with F=det([f(x),f(y),f(z)],[g(x),g(y),g(z)],[h(x),h(y),h(z)]). I will explain below how to greatly simplify your "simpler" version (let's call it w_D, for (weight system)_(Duzhin)). After my simplification the result is *almost* the sl(2) weight system, yet potentially still more general. I am not sure if it really is more general or if things can be simplified any further; at any case, it is intriguing.

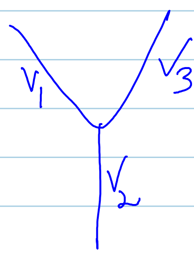
This is not consistent with what I said back in 2001, that your weight system is precisely equivalent to the sl(2) weight system. There may be several reasons for that, and I'm not sure which one it is -

1. It may be that your current description of the weight system is not quite identical to the one from 2001 and hence we are talking about different problems.
2. It may be that I've grown stupider since 2001 and what I could prove then I cannot prove now.
3. It may be that I've grown smarter since 2001 and the mistake I made back then I am not making now...

Anyway, let's move on to the math.

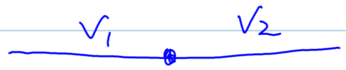
Firstly, we may as well denote f(x) by f_1, f(y) by f_2, g(z) by g_3, etc. So really your weight system is defined as follows -

- For Jacobi diagrams with n_1 internal edges and n_2 legs, choose n=n_1+n_2 vectors in R^3. Given one such Jacobi diagram D, create a sum over all bijections between the first n_1 vectors and the internal edges and the last n_2 vectors and the legs, where for each bijection the summand is the product over the trivalent vertices in D of the determinants of the matrix created from the 3 vectors assigned to the edges around that vertex. This sum is a scalar for each D, and you show that the resulting functional on Jacobi diagrams satisfies AS and IHX and hence it is a weight system.



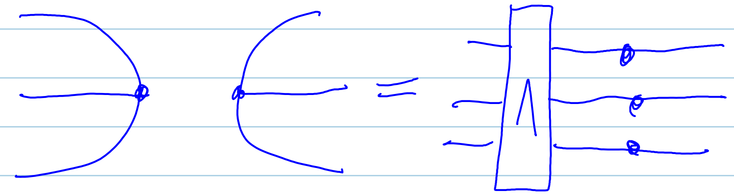
$$\longrightarrow \det\left(V_1 \,\middle|\, V_2 \,\middle|\, V_3\right)$$

First, extend this to graphs that are also allowed to have bivalent vertices, using the standard dot product of R^3 for the bivalent vertices:



$$\longrightarrow V_1 \cdot V_2 = \langle V_1, V_2 \rangle$$

The key point, which in itself is easy to prove, is that the extended w_D satisfies the relation,

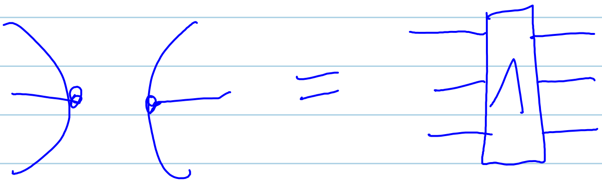
(1) 

Added June 7: Ouzhin comments that this is the "Gram det": $\det(v_1|v_2|v_3)\cdot\det(w_1|w_2|w_3) = \det(\langle v_i, w_j\rangle)$

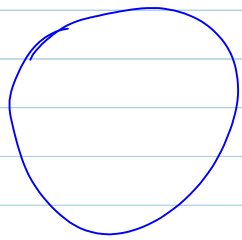
where \Lambda denotes anti-symmetrization. The proof is easy - both sides involve exactly 6 vectors in R^3, which get symmetrized. So it's just a finite computation. Note that the bivalent vertices are necessary so that the two sides would involve the same number of vectors.

Given this relation it is easy to reduce the computation of w_D to the computation of w_D restricted to circles with bivalent vertices and to arcs with bivalent vertices (and maybe one leftover trivalent vertices, which then vanishes for anti-symmetry reasons). Indeed, the relation allows you to get rid of trivalent vertices in pairs, until you are left with a diagram that has at most one trivalent vertex.

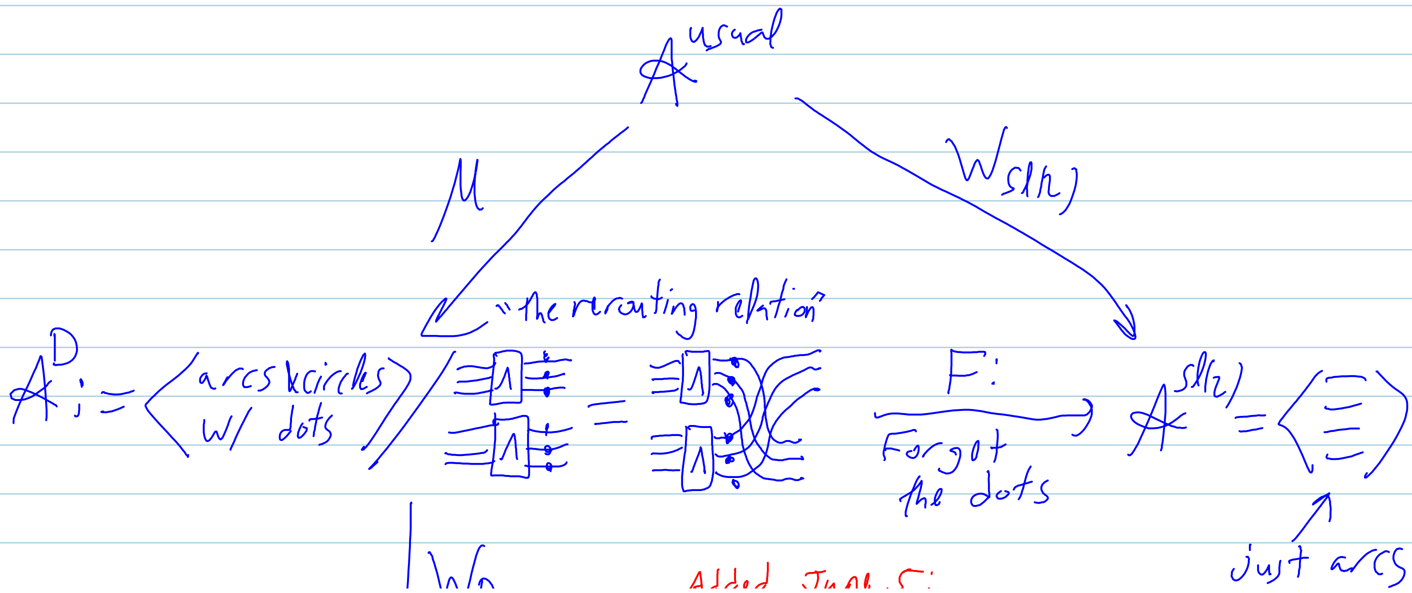
The sl(2) weight system satisfies a similar relation - the exact same thing, in fact, perhaps with a constant and without the dots:

(2) 

The sl(2) weight system also satisfies

(3)  = 3 (a closed loop = 3, as dim sl(2) = 3)

In summary, we can make the following diagram:



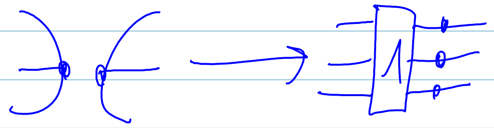
the dots

just arcs

$$\Big\downarrow W_D$$

$$\mathbb{R}$$

Added June 5: This picture works only for an even number of trivalent vertices; an amendment is necessary in the odd case.

Here μ is the map  ; it is easy to check that it is well-defined modulo the "re-routing relation" drawn above, and that it satisfies AS & IHX.

At this point, W_D stops being interesting on its own, as it anyway factors via a relatively simple map (μ) through a relatively simple space ($\mathcal{A}^D$). So let's understand $\mathcal{A}^D$.

Before applying the rerouting relation, $\mathcal{A}^D$ is a polynomial algebra in ∞ many variables - these are

$$\begin{cases} a_n: & \text{an arc with } n \text{ dots.} \\ c_n: & \text{a circle with } n \text{ dots.} \end{cases}$$

Modulo the rerouting relation, $\mathcal{A}^D$ is a quotient of a polynomial algebra in finitely many variables, as you can show that a_n & c_n satisfy some recursion relations.

But so far I did not write these recursion relations in full and I'm not sure how big A^0 really is.

In summary - The "det" version of your weight system has a simple combinatorial description using μ and A^0. It certainly contains the $sl(2)$ weight system and it seems to me that within its target space there is room for more, even if not so much more.

I'd be very interested to see what that "more" really is.

Best,

Dror.