Monads

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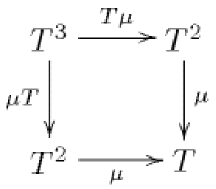
From http://en.wikipedia.org/wiki/Monad_%28category_theory%29:

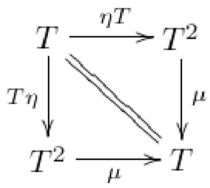
Formal definition

If C is a category, a **monad** on C consists of a functor $T\colon C \to C$ together with two natural transformations: $\eta\colon 1_C \to T$ (where 1_C denotes the identity functor on C) and $\mu\colon T^2 \to T$ (where T^2 is the functor $T \circ T$ from C to C). These are required to fulfill the following conditions (sometimes called coherence conditions):

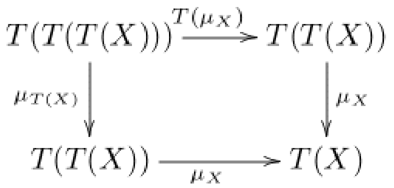
- $\mu \circ T\mu = \mu \circ \mu T$ (as natural transformations $T^3 \to T$);
- $\mu \circ T\eta = \mu \circ \eta T = 1_T$ (as natural transformations $T \to T$; here 1_T denotes the identity transformation from T to T).

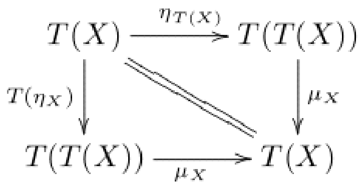
We can rewrite these conditions using following commutative diagrams:





See the article on natural transformations for the explanation of the notations $T\mu$ and μT, or see below the commutative diagrams not using these notions:





The first axiom is akin to the associativity in monoids, the second axiom to the existence of an identity element. Indeed, a monad on C can alternatively be defined as a monoid in the category $\mathbf{End}_C$ whose objects are the endofunctors of C and whose morphisms are the natural transformations between them, with the monoidal structure induced by the composition of endofunctors.

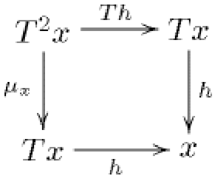
Algebras for a monad

[edit]

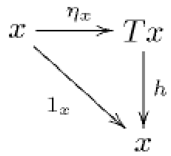
See also: F-algebra

Suppose that (T,η,μ) is a given monad on a category C.

A ***T*-algebra** (x,h) is an object x of C together with an arrow $h\colon Tx \to x$ of C called the **structure map** of the algebra such that the diagrams

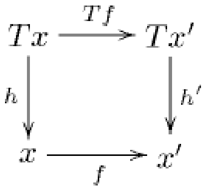


and



commute.

A morphism $f\colon (x,h) \to (x',h')$ of T-algebras is an arrow $f\colon x \to x'$ of C such that the diagram



commutes.

The category C^T of T-algebras and their morphisms is called the **Eilenberg-Moore category** or **category of (Eilenberg-Moore) algebras** of the monad T. The forgetful functor $C^T \to C$ has a left adjoint $C \to C^T$ taking x to the free algebra (Tx,μ_x).

Given the monad T, there exists another "canonical" category C_T called the **Kleisli category** of the monad T. This category is equivalent to the *category of free algebras* for the monad T, i. e. the full subcategory of C^T whose objects are of the form (Tx,μ_x), for x an object of C.