# Combinatorial Morse Theory

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11:17 AM

$$C_2 \underset{d_2}{\overset{U_1}{\rightleftarrows}} C_1 \underset{d_1}{\overset{U_0}{\rightleftarrows}} C_0$$

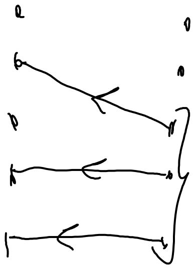
$(dU+Ud)^n = (dU)^n + (Ud)^n$

$U_0 = U_0\pi$

$(I - d_1U_0)\pi_0$

$(\pi_1 - d_2U_1)$ nilpotent  $(\pi_0 - d_1U_0)$ nilpotent

$\pi$: "project on the pivots".



$\pi_0$: project on these

$\Rightarrow \sim$

$1-\pi$ is homotopic to....

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$C_1 \underset{d}{\overset{u}{\rightleftarrows}} C_0$  $(1-du)$ is nilpotent.

$\Rightarrow (du)^{-1} = 1 + (1-du) + (1-du)^2 + \dots$

$\Rightarrow du(du)^{-1} = 1 = d(u(du)^{-1}) \Rightarrow d$ is surjective.

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$$\begin{tikzcd} C_1 \arrow[r, shift right, "d"'] & C_0 \arrow[l, shift right, "u"'] \arrow[l, bend right, dashed, "h"'] \arrow[loop right, "\pi"] \end{tikzcd}$$

Given $u\pi = u$, $\pi^2 = \pi$, $(1-du)\pi$ nilpotent.

~~Goal~~: $1-\pi \sim 1$; i.e. $\exists h$ s.t. $dh = \pi$

~~Proof~~ $\pi - du = N \Rightarrow du = \pi - N = \pi - N\pi = (1-N)\pi$

$\Rightarrow (1-N)^{-1} du = \pi$

Goal $H_0 \cong \ker \pi$

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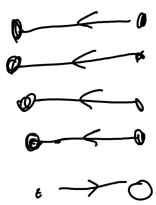
$(C_*, \partial)$  $\pi: C_* \to C_*$, $u: C_* \to C_{*+1}$ s.t.

$u = u\pi$ & $0 = \pi u$ & $\pi = \partial u$ & $\pi^2 = \pi \Rightarrow$

1. $\pi' = u\partial$ is a projection: $(u\partial)^2 = u\partial u\partial$
   $= u\pi\partial = u\partial$



2. $\pi'$ is disjoint from $\pi$:
   $\pi\pi' = \pi u\partial = 0$  $\pi'\pi = u\partial\pi = u\partial\partial u = 0$

3. $C_*^0 = (\pi + \pi')C_*$ is a subcomplex.

$$\partial(\pi+\pi') = \partial\pi + \partial u \partial = \partial u \partial = (\pi + u\partial)\partial = (\pi+\pi')\partial$$

4. $C_*^0$ is acyclic: On $C_*^0$, $\partial u + u\partial = 1$. Indeed,

$$\partial u + u \partial = \pi + \pi' = 1.$$

$\Longrightarrow$ $C_* / \mathrm{im}\,\pi + \mathrm{im}\,\pi'$ has same homology as $C_*$

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Now given $(C_*, \partial)$, $\pi : C_* \to C_*$ w/ $\pi^2 = \pi$,

$u : C_* \to C_{*+1}$ s.t. $u = u\pi$ & $\pi u = 0$

& $N = \pi - \partial u$ is nilpotent, we want a similar result.

$$\pi = \pi\partial h \Leftrightarrow \pi(\pi - \partial h) = 0 \quad \Bigg| \quad \begin{aligned} N\pi &= N \\ \pi N &= \pi - \pi\partial u \end{aligned}$$

$$\partial u = \pi - N = (1-N)\pi \Rightarrow (1-N)^{-1}\partial u = \pi$$

$$\pi\partial(u + uN + uN^2 \dots) = \pi \; ?$$

$$\pi\partial u (1-N)^{-1} = \pi \; ?$$

$$\pi\partial u (1-N)^{-1} = \pi(\pi - N)(1-N)^{-1} =$$

$$= \pi(1-N)(1-N)^{-1} = \pi \; !$$

1. $\pi' = h\partial$ is a projection: $\pi'^2 = h\partial h\partial$
$= h\pi\partial h\partial = h\pi\partial = h\partial$

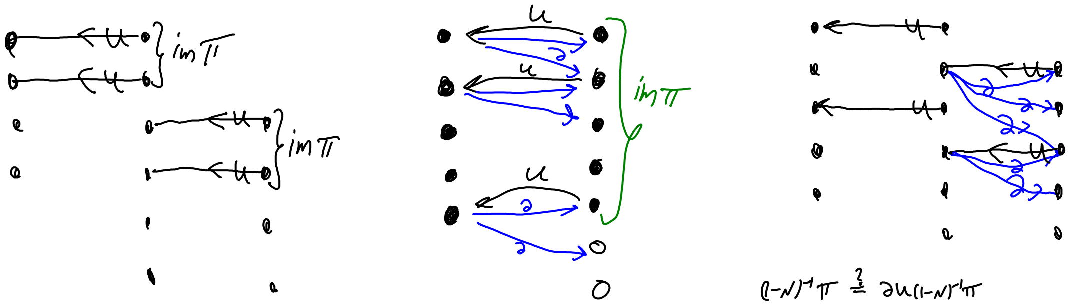
2. $\pi'$ is disjoint from $\pi$:
$\pi\pi' = \pi h\partial = \pi u(1-N)^{-1}\partial = 0$
$\pi'\pi = h\partial\pi = h\partial\pi\partial h =$

Probably a false start.

$$\boxed{\text{Aside: } u^2 = u\pi u = 0}$$

& $h\pi = \pi$

So with $h = u(1-N)^{-1}$ we have $\pi\partial h = \pi$




$$(1-N)^{-1}\pi \overset{?}{=} \partial u(1-N)^{-1}\pi$$

$$\partial u(1-N)^{-1}\pi = (\pi - N)(1-N)^{-1}\pi = (\pi - 1 + 1 - N)(1-N)^{-1}\pi$$

$$= (\pi - 1)(1-N)^{-1}\pi + \pi = (\pi-1)(\pi + N + N^2 + \dots) + \pi$$

$$\pi\partial u(1-N)^{-1}\pi = \pi$$

$$\begin{tikzcd} & \pi C \arrow[dl, "u(1-N)^{-1}\pi"'] \arrow[d, "\partial u(1-N)^{-1}\pi"] \\ {} & {} \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[dr, "\Pi\partial"'] & & C \arrow[dl, "\Pi"] \\ & \Pi C & \end{tikzcd}$$