Pensieve Header: Computing the Alexander polynomial following the Bigelow formalism. See also http://katlas.math.toronto.edu/drorbn/bbs/show?shot=Bigelow-110321-092650.jpg.

In[1]:=
```
<< KnotTheory`
```

```
Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
Read more at http://katlas.org/wiki/KnotTheory.
```

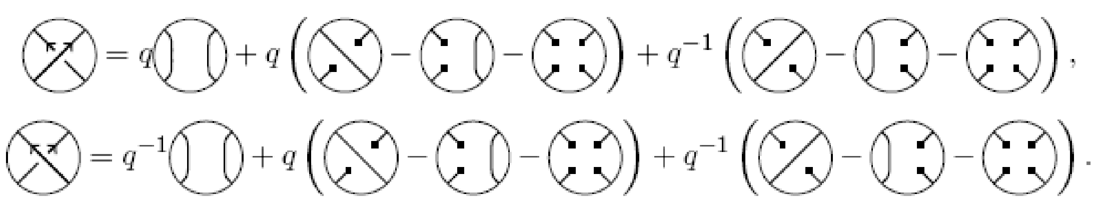
In[78]:=
```
EPD[pd_PD] := Module[
    {
     n = Length[First[Skeleton[pd]]]
    },
    (EPD @@ pd /. i_Integer /; i > n :> i + 1) /. {
       X[i_, j_, 1, k_] :> X[i, j, n + 1, k],
       X[i_, n, j_, 1] :> X[i, n, j, n + 1],
       X[i_, 1, j_, n] :> X[i, n + 1, j, n]
      } /. {
      X[i_, j_, k_, l_] /; PositiveQ[X[i, j, k, l]] :> Xp[i, j, k, l],
      X[i_, j_, k_, l_] /; NegativeQ[X[i, j, k, l]] :> Xm[i, j, k, l]
     }
   ];
EPD[K_Knot] := EPD[PD[K]];
```

In[80]:=
```
EPD[Knot[4, 1]]
```

Out[80]=
```
EPD[Xp[4, 2, 5, 1], Xp[8, 6, 9, 5], Xm[6, 3, 7, 4], Xm[2, 7, 3, 8]]
```

In[51]:=
```
K = Knot[4, 1];
n = Crossings[K];
epd = EPD[Knot[4, 1]]
```

Out[53]=
```
EPD[Xp[4, 2, 5, 1], Xp[8, 6, 9, 5], Xm[6, 3, 7, 4], Xm[2, 7, 3, 8]]
```



In[54]:=
```
t1 = Expand[Times @@ epd /. {
    Xp[i_, j_, k_, l_] :> (
      q OP[l, k] OP[i, j]
       + q (OP[i, k] DE[j] DE[l] - OP[i, j] DE[l] DE[k] - DE[i] DE[j] DE[k] DE[l])
       + 1 / q (OP[l, j] DE[i] DE[k] - OP[l, k] DE[j] DE[i] - DE[i] DE[j] DE[k] DE[l])
     ),
    Xm[i_, j_, k_, l_] :> (
      1 / q OP[i, l] OP[j, k]
       + q (OP[j, l] DE[i] DE[k] - OP[j, k] DE[i] DE[l] - DE[i] DE[j] DE[k] DE[l])
       + 1 / q (OP[i, k] DE[j] DE[l] - OP[i, l] DE[j] DE[k] - DE[i] DE[j] DE[k] DE[l])
     )
   }]
```

A very large output was generated. Here is a sample of it:

Out[54]= 6 DE[1] DE[2]^2 DE[3]^2 DE[4]^2 DE[5]^2 DE[6]^2 DE[7]^2 DE[8]^2 DE[9] + <<3238>> + q^2 DE[1] DE[2]^2 DE[3]^2 DE[4] DE[5] DE[6] OP[4, 5] OP[6, 7] OP[7, 8] OP[8, 9]

Show Less | Show More | Show Full Output | Set Size Limit...

In[55]:=
```
(t1 //. OP[i_, j_] OP[j_, k_] :> OP[i, k]) //. P[i_, j_] DE[j_] :> DE[i] /. {
  DE[_]^2 → 1, P[_, _]^2 → 0, P[i_, i_] → 0
 }
```

A very large output was generated. Here is a sample of it:

Out[55]= 6 DE[1] DE[9] + $\frac{\text{DE[1] DE[9]}}{q^4}$ + $\frac{4\,\text{DE[1] DE[9]}}{q^2}$ + 4 q^2 DE[1] DE[9] + <<2826>> + q^4 DE[1] DE[4] DE[7] DE[8] OP[7, 4] OP[8, 9] - q^2 DE[2] DE[4] DE[7] DE[8] OP[1, 2] OP[7, 4] OP[8, 9] + q^2 DE[4] DE[5] DE[7] DE[8] OP[1, 5] OP[7, 4] OP[8, 9] - q^4 DE[1] DE[5] DE[7] DE[8] OP[7, 5] OP[8, 9]

Show Less | Show More | Show Full Output | Set Size Limit...

In[56]:=
```
Alexander[K][t]
```

Out[56]= $3 - \frac{1}{t} - t$

```
In[57]:= BigAlex[EPD[], _, r_] := r;
BigAlex[epd_EPD, done_, r_] := Module[{pos},
   pos = First[Ordering[Length[Complement[List @@ #, done]] & /@ epd]];
   BigAlex[
    Delete[epd, pos],
    Union[done, List @@ epd[[pos]]],
    Expand[r * epd[[pos]] /. {
          Xp[i_, j_, k_, l_] :> (
            q OP[l, k] OP[i, j]
             + q (OP[i, k] DE[j] DE[l] - OP[i, j] DE[l] DE[k] - DE[i] DE[j] DE[k] DE[l])
             + 1 / q
              (OP[l, j] DE[i] DE[k] - OP[l, k] DE[j] DE[i] - DE[i] DE[j] DE[k] DE[l])
           ),
          Xm[i_, j_, k_, l_] :> (
            1 / q OP[i, l] OP[j, k]
             + q (OP[j, l] DE[i] DE[k] - OP[j, k] DE[i] DE[l] - DE[i] DE[j] DE[k] DE[l])
             + 1 / q
              (OP[i, k] DE[j] DE[l] - OP[i, l] DE[j] DE[k] - DE[i] DE[j] DE[k] DE[l])
           )
         }] //.
       OP[i_, j_] OP[j_, k_] :> OP[i, k] //.
      {
       OP[i_, j_] DE[j_] :> DE[i],
       DE[i_] OP[i_, j_] :> DE[j]
      } /.
     {
      DE[_]^2 → 1, OP[i_, i_] → 0
     }
   ]
  ];
BigAlex[epd_EPD] := BigAlex[epd, {}, 1]

In[60]:= epd

Out[60]= EPD[Xp[4, 2, 5, 1], Xp[8, 6, 9, 5], Xm[6, 3, 7, 4], Xm[2, 7, 3, 8]]

In[61]:= done = {}

Out[61]= {}

In[62]:= Length[Complement[List @@ #, done]] & /@ epd

Out[62]= EPD[4, 4, 4, 4]

In[63]:= pos = First[Ordering[Length[Complement[List @@ #, done]] & /@ epd]]

Out[63]= 1

In[64]:= BigAlex[epd]
```

Out[64]= $-OP[1, 9] - \frac{OP[1, 9]}{q^4} + \frac{3\, OP[1, 9]}{q^2}$

In[65]:=
```
EPD[Knot[4, 1]]
```

Out[65]= EPD[Xp[4, 2, 5, 1], Xp[8, 6, 9, 5], Xm[6, 3, 7, 4], Xm[2, 7, 3, 8]]

In[66]:=
```
BigAlex[EPD[Knot[10, 42]]]
```

Out[66]=
$$27\,\mathrm{OP}[1,21]-\frac{\mathrm{OP}[1,21]}{q^6}+\frac{7\,\mathrm{OP}[1,21]}{q^4}-\frac{19\,\mathrm{OP}[1,21]}{q^2}-19\,q^2\,\mathrm{OP}[1,21]+7\,q^4\,\mathrm{OP}[1,21]-q^6\,\mathrm{OP}[1,21]$$

In[21]:=
```
Expand[OP[1, 21] Alexander[Knot[10, 42]][q^2] / q^0]
```

Out[21]=
$$27\,\mathrm{OP}[1,21]-\frac{\mathrm{OP}[1,21]}{q^6}+\frac{7\,\mathrm{OP}[1,21]}{q^4}-\frac{19\,\mathrm{OP}[1,21]}{q^2}-19\,q^2\,\mathrm{OP}[1,21]+7\,q^4\,\mathrm{OP}[1,21]-q^6\,\mathrm{OP}[1,21]$$

In[22]:=
```
BigAlex[EPD[Xp[2, 4, 3, 1], Xm[3, 4, 6, 5]]]
```

Out[22]= OP[1, 5] OP[2, 6]

In[23]:=
```
t1 = BigAlex[EPD[Xm[3, 1, 2, 4], Xp[6, 5, 3, 4]]]
```

Out[23]=
$$-2\,\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DE}[5]\,\mathrm{DE}[6]+\mathrm{DE}[5]\,\mathrm{DE}[6]\,\mathrm{OP}[1,2]+\mathrm{DE}[2]\,\mathrm{DE}[6]\,\mathrm{OP}[1,5]+\mathrm{DE}[1]\,\mathrm{DE}[5]\,\mathrm{OP}[6,2]+\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{OP}[6,5]-\mathrm{OP}[1,2]\,\mathrm{OP}[6,5]$$

In[24]:=
```
t2 = Expand[OP[1, 5] OP[6, 2] - t1 /. OP[i_, j_] :> DP[i, j] + DE[i] DE[j]]
```

Out[24]= DP[1, 5] DP[6, 2] + DP[1, 2] DP[6, 5]

In[25]:=
```
t2 //. DP[is__] DP[js__] :> DP[is, js] //. dp_DP :> Signature[dp] Sort[dp]
```

Out[25]= 0

In[26]:=
```
r31 = EPD[Xp[7, 9, 6, 1], Xp[3, 8, 7, 2], Xp[8, 4, 5, 9]];
r32 = EPD[Xp[3, 4, 7, 9], Xp[7, 5, 6, 8], Xp[2, 9, 8, 1]];
{BigAlex[r31], BigAlex[r32]}
```

Out[28]=
$$\Big\{2\,q\,\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[5]\,\mathrm{DE}[6]+2\,q^3\,\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[5]\,\mathrm{DE}[6]-$$
$$\frac{\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[5]\,\mathrm{DE}[6]\,\mathrm{OP}[1,4]}{q^3}+\frac{\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[6]\,\mathrm{OP}[1,5]}{q^3}-$$
$$\frac{2\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[6]\,\mathrm{OP}[1,5]}{q}-\frac{\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[5]\,\mathrm{OP}[1,6]}{q^3}+$$
$$\frac{2\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[5]\,\mathrm{OP}[1,6]}{q}-q^3\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[5]\,\mathrm{OP}[1,6]-$$
$$q\,\mathrm{DE}[1]\,\mathrm{DE}[3]\,\mathrm{DE}[5]\,\mathrm{DE}[6]\,\mathrm{OP}[2,4]+\frac{\mathrm{DE}[3]\,\mathrm{DE}[6]\,\mathrm{OP}[1,5]\,\mathrm{OP}[2,4]}{q}-$$
$$\frac{\mathrm{DE}[3]\,\mathrm{DE}[5]\,\mathrm{OP}[1,6]\,\mathrm{OP}[2,4]}{q}+q\,\mathrm{DE}[3]\,\mathrm{DE}[5]\,\mathrm{OP}[1,6]\,\mathrm{OP}[2,4]-$$
$$\frac{\mathrm{DE}[1]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[6]\,\mathrm{OP}[2,5]}{q}+q\,\mathrm{DE}[1]\,\mathrm{DE}[3]\,\mathrm{DE}[4]\,\mathrm{DE}[6]\,\mathrm{OP}[2,5]-$$

$$
\begin{aligned}
&q\,DE[3]\,DE[4]\,OP[1,6]\,OP[2,5] + \frac{DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6]}{q} - \\
&q\,DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6] - q^3\,DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6] - \\
&q\,DE[1]\,DE[2]\,DE[5]\,DE[6]\,OP[3,4] + q^3\,DE[1]\,DE[2]\,DE[5]\,DE[6]\,OP[3,4] + \\
&q\,DE[2]\,DE[5]\,OP[1,6]\,OP[3,4] - q^3\,DE[2]\,DE[5]\,OP[1,6]\,OP[3,4] - \\
&q^3\,DE[1]\,DE[6]\,OP[2,5]\,OP[3,4] + q^3\,OP[1,6]\,OP[2,5]\,OP[3,4] + \\
&q\,DE[1]\,DE[2]\,DE[4]\,DE[6]\,OP[3,5] - 2\,q^3\,DE[1]\,DE[2]\,DE[4]\,DE[6]\,OP[3,5] - \\
&q\,DE[2]\,DE[4]\,OP[1,6]\,OP[3,5] + q^3\,DE[2]\,DE[4]\,OP[1,6]\,OP[3,5] + \\
&q^3\,DE[1]\,DE[4]\,OP[2,6]\,OP[3,5] - q\,DE[1]\,DE[2]\,DE[4]\,DE[5]\,OP[3,6], \\
&2\,q\,DE[1]\,DE[2]\,DE[3]\,DE[4]\,DE[5]\,DE[6] + 2\,q^3\,DE[1]\,DE[2]\,DE[3]\,DE[4]\,DE[5]\,DE[6] - \\
&\frac{DE[2]\,DE[3]\,DE[5]\,DE[6]\,OP[1,4]}{q^3} + \frac{DE[2]\,DE[3]\,DE[4]\,DE[6]\,OP[1,5]}{q^3} - \\
&\frac{2\,DE[2]\,DE[3]\,DE[4]\,DE[6]\,OP[1,5]}{q} - \frac{DE[2]\,DE[3]\,DE[4]\,DE[5]\,OP[1,6]}{q^3} + \\
&\frac{2\,DE[2]\,DE[3]\,DE[4]\,DE[5]\,OP[1,6]}{q} - q^3\,DE[2]\,DE[3]\,DE[4]\,DE[5]\,OP[1,6] - \\
&q\,DE[1]\,DE[3]\,DE[5]\,DE[6]\,OP[2,4] + \frac{DE[3]\,DE[6]\,OP[1,5]\,OP[2,4]}{q} - \\
&\frac{DE[3]\,DE[5]\,OP[1,6]\,OP[2,4]}{q} + q\,DE[3]\,DE[5]\,OP[1,6]\,OP[2,4] - \\
&\frac{DE[1]\,DE[3]\,DE[4]\,DE[6]\,OP[2,5]}{q} + q\,DE[1]\,DE[3]\,DE[4]\,DE[6]\,OP[2,5] - \\
&q\,DE[3]\,DE[4]\,OP[1,6]\,OP[2,5] + \frac{DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6]}{q} - \\
&q\,DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6] - q^3\,DE[1]\,DE[3]\,DE[4]\,DE[5]\,OP[2,6] - \\
&q\,DE[1]\,DE[2]\,DE[5]\,DE[6]\,OP[3,4] + q^3\,DE[1]\,DE[2]\,DE[5]\,DE[6]\,OP[3,4] + \\
&q\,DE[2]\,DE[5]\,OP[1,6]\,OP[3,4] - q^3\,DE[2]\,DE[5]\,OP[1,6]\,OP[3,4] - \\
&q^3\,DE[1]\,DE[6]\,OP[2,5]\,OP[3,4] + q^3\,OP[1,6]\,OP[2,5]\,OP[3,4] + \\
&q\,DE[1]\,DE[2]\,DE[4]\,DE[6]\,OP[3,5] - 2\,q^3\,DE[1]\,DE[2]\,DE[4]\,DE[6]\,OP[3,5] - \\
&q\,DE[2]\,DE[4]\,OP[1,6]\,OP[3,5] + q^3\,DE[2]\,DE[4]\,OP[1,6]\,OP[3,5] + \\
&q^3\,DE[1]\,DE[4]\,OP[2,6]\,OP[3,5] - q\,DE[1]\,DE[2]\,DE[4]\,DE[5]\,OP[3,6] \}
\end{aligned}
$$

In[29]:= **BigAlex[r31] - BigAlex[r32]**

Out[29]= 0

In[30]:=
```
res4 = BigAlex /@ {
   EPD[
    Xm[19, 1, 20, 2], Xp[11, 3, 12, 2], Xp[3, 30, 4, 29], Xm[4, 21, 5, 22],
    Xp[6, 23, 7, 22], Xm[7, 28, 8, 29], Xm[12, 8, 13, 9], Xp[18, 10, 19, 9],
    Xm[27, 13, 28, 14], Xp[23, 15, 24, 14], Xm[24, 16, 25, 17], Xp[26, 18, 27, 17]
   ],
   EPD[
    Xp[1, 28, 2, 27], Xm[2, 23, 3, 24], Xm[17, 3, 18, 4], Xp[13, 5, 14, 4],
    Xm[14, 6, 15, 7], Xp[16, 8, 17, 7], Xp[8, 25, 9, 24], Xm[9, 26, 10, 27],
    Xm[29, 11, 30, 12], Xp[21, 13, 22, 12], Xm[22, 18, 23, 19], Xp[28, 20, 29, 19]
   ]
  }
```

Out[30]= $Aborted

```
{1, -1}.res4
```

A very large output was generated. Here is a sample of it:

$$2\,\mathrm{DE}[6]\,\mathrm{DE}[10]\,\mathrm{DE}[11]\,\mathrm{DE}[15]\,\mathrm{DE}[16]\,\mathrm{DE}[20]\,\mathrm{DE}[21]\,\mathrm{DE}[25]\,\mathrm{DE}[26]\,\mathrm{DE}[30]\,\mathrm{OP}[1,5] + \ll 4081 \gg + \frac{\mathrm{DE}[1]\,\mathrm{DE}[5]\,\mathrm{DE}[10]\,\mathrm{DE}[11]\,\mathrm{DE}[16]\,\mathrm{DE}[20]\,\mathrm{OP}[6,15]\,\mathrm{OP}[21,25]\,\mathrm{OP}[26,30]}{q^2}$$

Show Less | Show More | Show Full Output | Set Size Limit...

```
Expand[
   {1, -1}.res4 /. OP[i_, j_] ⧴ DP[i, j] + DE[i] DE[j]
  ] //. DP[is__] DP[js__] ⧴ DP[is, js] //. dp_DP ⧴ Signature[dp] Sort[dp]
```

0

In[33]:=
```
Expand[BigAlex[EPD[Xp[1, 2, 3, 4]]] /. OP[i_, j_] ⧴ DP[i, j] + DE[i] DE[j]
  ] //. DP[is__] DP[js__] ⧴ DP[is, js] //. dp_DP ⧴ Signature[dp] Sort[dp]
```

Out[33]=
$$-\frac{\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DE}[3]\,\mathrm{DE}[4]}{q} + q\,\mathrm{DE}[2]\,\mathrm{DE}[4]\,\mathrm{DP}[1,3] - \frac{\mathrm{DE}[1]\,\mathrm{DE}[3]\,\mathrm{DP}[2,4]}{q} + \frac{\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DP}[3,4]}{q} - q\,\mathrm{DE}[1]\,\mathrm{DE}[2]\,\mathrm{DP}[3,4] - q\,\mathrm{DP}[1,2,3,4]$$

In[67]:=
```
epd = EPD[PD[Link["L6a4"]]]
```

Out[67]=
```
EPD[Xm[6, 1, 7, 2], Xp[12, 8, 9, 7], Xp[4, 12, 13, 11],
 Xm[10, 5, 11, 6], Xp[8, 4, 5, 3], Xm[2, 9, 3, 10]]
```

In[68]:=
```
BigAlex[epd] // Simplify
```

Out[68]=
$$\frac{\left(-1+q^2\right)^4\,\mathrm{OP}[1,13]}{q^4}$$

In[70]:=
```
MultivariableAlexander[Link["L6a4"]][t] /. t[i_] → q^2
```

Out[70]=

$$\frac{\left(-1+q^2\right)^3}{\left(q^2\right)^{3/2}}$$

In[76]:=
```
pd = PD[Link["L9n17"]]
```

Out[76]=
```
PD[X[8, 1, 9, 2], X[2, 9, 3, 10], X[10, 3, 11, 4], X[7, 14, 8, 15], X[13, 18, 14, 7],
 X[17, 1, 18, 6], X[16, 11, 17, 12], X[5, 12, 6, 13], X[4, 16, 5, 15]]
```

In[77]:=
```
{
 n = Length[First[Skeleton[pd]]],
 m = Crossings[pd]
}
```

Out[77]=
```
{6, 9}
```

In[85]:=
```
EPD[PD[Link["L9n19"]]]
```

Out[85]=
```
EPD[Xm[11, 1, 12, 2], Xm[3, 13, 4, 14], Xm[19, 5, 10, 6],
 Xm[6, 10, 7, 11], Xp[17, 13, 18, 12], Xm[7, 15, 8, 16],
 Xm[14, 4, 15, 5], Xm[16, 8, 17, 9], Xm[2, 18, 3, 19]]
```

In[86]:=
```
BigAlex[EPD[PD[Link["L9n19"]]]] // Factor
```

Out[86]=

$$-\frac{(-1+q)\,(1+q)\,\left(1+q^2\right)^2\,\left(1-q^2+q^4\right)\,\mathrm{OP}[1,9]}{q^5}$$

In[89]:=
```
MultivariableAlexander[Link["L9n19"]][t] /. t[i_] → q^2 // Factor
```

Out[89]=

$$-\frac{\left(1+q^2\right)^2\,\left(1-q^2+q^4\right)}{q^4}$$