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Criterion for Quadraticity of the Associated Graded Algebra of an Augmented Algebra

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Abstract

If an augmented algebra A over $\mathbb{Q}$ is filtered by powers of its augmentation ideal I, the associated graded algebra $gr_I A$ need not in general be quadratic: although it is generated in degree 1, its relations may not be generated by homogeneous relations of degree 2. In this paper we give a criterion which is equivalent to $gr_I A$ being quadratic, generalizing a result of [Hutchings] in which the algebra was the group ring of the pure braid group and the ideal was the augmentation ideal. We apply this criterion to the pure virtual braid group (also known as the quasi-triangular group), and show that the corresponding associated graded ring is quadratic.

[Alt. title: The Pure Virtual Braid Group is Quadratic] I like this better.

Contents

1 Introduction

This paper will ultimately be concerned with the pure virtual braid groups $\boldsymbol{PvB_n}$, for all $n \in \mathbb{N}$, generated by symbols R_{ij}, $1 \leq i \neq j \leq n$, with relations the Reidemeister III moves (or quantum Yang-Baxter relations) and certain commutativities:

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \tag{1}$$
$$R_{ij}R_{kl} = R_{kl}R_{ij}, \tag{2}$$

with i, j, k, l distinct. This group is referred to as the quasi-triangular group $\boldsymbol{QTr_n}$ in [BarEnEtRa]. We will also be concerned with the related algebra $\mathfrak{pvb}_n$, generated by symbols r_{ij}, $1 \leq i \neq j \leq n$, with relations the '6-term' (or 6T) relations, and related commutativities:

$$\begin{aligned} y_{ijk} := [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0, \\ c_{ij}^{kl} := [r_{ij}, r_{kl}] = 0 \end{aligned} \tag{3}$$

with i, j, k, l distinct. This algebra is the universal enveloping algebra of the quasi-triangular Lie algebra $\mathfrak{qtr}_n$ in [BarEnEtRa].

We will show that PvB_n is a 'quadratic group', in the sense that if its rational group ring $\mathbb{Q}PvB_n$ is filtered by powers of the augmentation ideal I, the associated graded ring $grPvB_n$ is a quadratic algebra: i.e., a graded algebra generated in degree 1, with relations generated by homogeneous relations of degree 2. We note that, in different language, this is the statement that PvB_n has a universal finite-type invariant, which takes values in the algebra $\mathfrak{pvb}_n$.

In [Hutchings], a criterion was given for the quadraticity of the pure braid group. The proof relied on the geometry of braids embedded in $\mathbf{R}^3$. In order to generalize this criterion to all finitely presented groups, we developed an algebraic proof of the criterion. This proof turned out not to rely on the existence of an underlying group, and applies instead to algebras over $\mathbb{Q}$, filtered by powers of an augmentation ideal I. Indeed, this criterion arguably lives naturally in an even broader context, such as perhaps augmented algebras over an operad (or the related 'circuit algebras' of [BN-WKO]). In the first part of this paper we state this generalized criterion as it applies to any augmented algebra and prove that, if satisfied, it implies that the associated graded algebra is quadratic (see Theorem 2). In the second part of the paper, we specialize to PvB_n. We present a basis for the quadratic dual algebra $\mathfrak{pvb}_n^!$, and use this basis to prove that PvB_n satisfies the generalized Hutchings Criterion. It follows that PvB_n is quadratic. As pointed out in section 8.5 of [BarEnEtRa], this implies that $H^*(PvB_n) \cong \mathfrak{pvb}_n^!$ as algebras.

1.1 Overview of the Hutchings Criterion

1.1.1 Group Theoretic Background

Since the classic setting of the Hutchings criterion is that of group rings, we identify the attributes of group rings which we rely on and will want to see preserved in our generalized context. We recall the follow basic fact:

Proposition 1 (See [MKS], s. 5.15). *If G if given by the short exact sequence*

$$1 \to N \to FG \to G \to 1$$

"F things" are not used until later, so perhaps they should be introduced later?

where FG is a free group generated by symbols $\{g_p :\ p \in P\}$ and N is a normal subgroup of FG generated by the set $\{r_q :\ q \in Q\}$, then the rational group ring of G is given by the exact sequence

Is there a reason to distinguish $p \leftrightarrow g_p$ and $q \leftrightarrow r_q$?

$$0 \to (N-1) \to \mathbb{Q}FG \to \mathbb{Q}G \to 0$$

where $(N-1)$ is the two-sided ideal in $\mathbb{Q}FG$ generated by $\{(r_q - 1) :\ q \in Q\}$.

We can clearly restrict the second exact sequence to the exact sequence

$$0 \to (N-1) \to I_{FG} \to I_G \to 0 \tag{4}$$

where I_{FG} and I_G are the augmentation ideals of $\mathbb{Q}FG$ and $\mathbb{Q}G$ respectively.

1.1.2 Generalized Algebraic Setting

By analogy with the above group case, we take A to be an augmented unital algebra over $\mathbb{Q}$ with 2-sided augmentation ideal I_A, and F to be the free algebra over $\mathbb{Q}$ with the same generating set as A, with 2-sided augmentation ideal I_F. In particular we assume an exact sequence:

$$0 \longrightarrow I_A \longrightarrow A \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0$$

By analogy with the ideal $(N-1)$ in the group context, we let $M \subseteq I_F \subseteq F$ be a 2-sided ideal such that:

$$0 \longrightarrow M \longrightarrow F \longrightarrow A \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow I_F \longrightarrow I_A \longrightarrow 0$$

are exact. It will be important in what follows that I_F is a free algebra, as well as a free F-module.

Finally we will generally assume that $M \subseteq I_F^2$. In the group context, this follows if each relator of the group has total degree zero in each generator. Although this requirement may not be necessary if one were to develop the Hutchings Criterion in complete generality, it does simplify things and at least presents no problem in many applications, including the specific application we have in mind (i.e. quadraticity of PvB_n).

explain?

1.1.3 The Two Canonically Associated Graded Algebras

A is filtered by powers of I_A. One associates to the pair (A, I_A) two canonically defined graded algebras. Firstly, there is the associated graded of the above filtration, denoted $\boldsymbol{gr_I A}$. We have $gr_I A \cong \bigoplus_m I_A^m/I_A^{m+1}$. It is clear that $gr_I A$ is generated as an algebra by its degree one piece $\boldsymbol{V} := I_A/I_A^2$, a vector space over $\mathbb{Q}$.

Secondly, there is a quadratic algebra U, which we will sometimes refer to as the associated quadratic algebra. U is generated as an algebra over $\mathbb{Q}$ by V; and, if TV is the rational tensor algebra over V (with tensor products over $\mathbb{Q}$), we let $\langle\mathfrak{R}\rangle$ be the two-sided ideal in TV generated by the vector subspace $\mathfrak{R} \subseteq V \otimes_{\mathbb{Q}} V$ of degree two relations of $gr_I A$: i.e., $\mathfrak{R} := ker(\mu : I_A/I_A^2 \otimes_{\mathbb{Q}} I_A/I_A^2 \to I_A^2/I_A^3)$, where μ is the multiplication in $gr_I A$ induced from multiplication in I_A. Then we define $U := TV/\langle\mathfrak{R}\rangle$. We will denote by U_m the m-th graded piece of U.

We note that since U has the same generators and the same quadratic relations as $gr_I A$, there is always a surjection $U \twoheadrightarrow gr_I A$. Quadraticity of $gr_I A$ is thus equivalent to the fact that this surjection is an isomorphism $U_m \cong I_A^m/I_A^{m+1}$, for all m. We will often use this alternative definition of quadraticity.

The premise of this section is a bit weird. U is given by generators & relations.

1.1.4 Generators and Relations for the Associated Quadratic Algebra

where? \cdot

We will see that $V^{\otimes_{\mathbb{Q}} m} \cong I_A^{\otimes_A m}/I_A.I_A^{\otimes_A m}$, or equivalently that if, in the following sequence, we define ι as the inclusion and F^0 as the projection to $coker\ \iota$, the sequence is exact:

imprecise. The exactness of (5) is tautological. What isn't trivial is that $coker\ \iota \cong V^{\otimes_{\mathbb{Q}} m}$

$$0 \longrightarrow I_A.I_A^{\otimes_A m} \xrightarrow{\iota} I_A^{\otimes_A m} \xrightarrow{F^0} coker\ \iota = V^{\otimes_{\mathbb{Q}} m} \longrightarrow 0 \tag{5}$$

By definition, the spaces $V^{\otimes_{\mathbb{Q}} m}$ generate the U_m.

$$\frac{I_A}{I_A^2} \otimes_{\mathbb{Q}} \frac{I_A}{I_A^2} \otimes_{\mathbb{Q}} \cdots \otimes_{\mathbb{Q}} \frac{I_A}{I_A^2} \longleftrightarrow \frac{I_A \otimes_A I_A \otimes \cdots \otimes I_A}{I_A(I_A \otimes_A I_A \cdots \otimes I_A)}$$

$$\longleftarrow a_1 b \otimes a_2 - a_1 \otimes b a_2 = a_1(b-1)\otimes a_2 - a_1 \otimes (b-1) a_2$$

We get a space of free generators for the relations in U as follows. Define:

$$\mathfrak{R}_{m,j} := V^{\otimes_{\mathbb{Q}} j} \otimes_{\mathbb{Q}} \mathfrak{R} \otimes_{\mathbb{Q}} V^{\otimes_{\mathbb{Q}}(m-j-2)}$$

Then a space of free generators for the degree m relations in U is given by:

$$\mathfrak{R}_m = \bigoplus_{j=1}^{m-2} \mathfrak{R}_{m,j}$$

1.1.5 Generators and Relations for I^m

We introduce a 'product' $\mu_i : I_A^{\otimes_A m} \to I_A^{\otimes_A m-1}$ which is multiplication of components i and $i+1$ in the tensor product. Since we are tensoring over A, $\mu_i = \mu_j$ for all i, j, so we often refer to the product as simply μ_A.

$I_A^{\otimes_A m}$ can viewed as a space of generators for I_A^m, for any m, because we have the surjection $\mu_A^{m-1} : I_A^{\otimes_A m} \twoheadrightarrow I^m$.

By analogy with $\mathfrak{R}$, we define $R^A := ker(\mu_A : I_A \otimes_A I_A \to I_A^2)$, and by analogy with $\mathfrak{R}_{m,j}$ and $\mathfrak{R}_m$, we define:

$$R^A_{m,j} := I_A^{\otimes_A j} \otimes_A R^A \otimes_A I_A^{\otimes_A (m-j-2)}$$

and

$$R^A_m := \bigoplus_j R^A_{m,j} \tag{6}$$

there is an obvious map $\partial_A : R^A_m \to I_A^{\otimes_A m}$

$\partial_A R^A_m \subset ker\,\mu_A$

By construction, the ~~R^A_m are in $ker\ \mu_A$~~, and hence give at least some of the relations for I_A^m in this presentation; as we will see, if the Hutchings Criterion (explained in Theorem 2 below) is met, R^A_m gives all the relations in this presentation; that is, ~~R^A_m surjects to $ker\,\mu_A$~~ $\partial_A R^A_m = ker\,\mu_A$.

In the braid case, this is a preliminary to the Hutchings theory, not a consequence thereof

1.1.6 The Hutchings Criterion

We are now in a position to state our criterion for quadraticity of A:

~~**Theorem 1** (Hutchings Criterion).~~ *With notation as above, we have the following exact commutative diagram:*

exact in what sense?

F^{Syz} or F_{Syz}?

Improve spacing for F^{Syz} & ∂^{Ind}

$$\begin{tikzcd}
ker\ \partial^{Ind} \arrow[dr] & & ker\ \partial_A \arrow[ll, "F^{Syz}"'] \arrow[dl] \\
& R^A_m \arrow[dl, "\partial_A"'] \arrow[dr, "\partial^{Ind}"] & \\
I_A^{\otimes_A m} \arrow[rr, "F^0"] & & (I_A/I_A^2)^{\otimes_{\mathbb{Q}} m}
\end{tikzcd}$$

This diagram is a preliminary to the theorem, not yet the theorem itself.

R^A_m = (TF T&T relations (deg 2 topology))

(m-singular braids) → (m-chord diagrams)

F_{Syz} is an inclusion

where ∂_A consists of the summand-wise inclusions, F^0 is defined in (5), ∂^{Ind} and F_{Syz} are the maps induced by commutativity of the lower triangle.

Then A is quadratic if and only if F_{Syz} is surjective. theorem 1 starts here.

This generalizes a result first obtained in [Hutchings], where A was the group ring of the pure braid group (see also [BNStoil]). very cryptic

In the cases of interest to us, we can get more precise information about R^A_m, and hence also *ker* ∂^{Ind}. Specifically,

Proposition 2. *Under the conditions of the previous theorem, let us take $\{y_q : q \in Q\}$ to be a minimal set of generators for M as a two-sided F-module. Suppose the $\{y_q + I^3_F : q \in Q\}$ are linearly independent in $(M + I^3_F)/I^3_F$. Then there is an isomorphism F^1:*

$$R^A_m \cong \mathfrak{R}_m$$

as vector spaces over $\mathbb{Q}$. Moreover, $\partial^{Ind} = \partial_U \circ F^1$, where ∂_U is the summand-wise inclusion of $\mathfrak{R}_m$ into $V^{\otimes_{\mathbb{Q}} m}$.

1.2 How the Hutchings Criterion is Useful

Note that, under the conditions of the last Proposition, the elements of *ker* $\partial_U \cong$ ker ∂^{Ind} correspond to relations among relations in the associated quadratic algebra U, which may be called infinitesimal syzygies. It is often the case that the syzygies of a quadratic algebra can be determined quite explicitly, using quadratic duality. Essentially, if the quadratic algebra U is Koszul, then the syzygies are generated by $U^{!3}$ (i.e. the degree 3 part of the quadratic dual $U^!$ of U).

Similarly, the elements of *ker* ∂_A are relations among the elements of R^A_m, which we may call global syzygies. The Hutchings Criterion asserts that if F_{Syz} maps *ker* ∂_A onto *ker* ∂_U, then $I^m_A/I^{m+1}_A \cong U_m$.

In the context of PvB_n, if we take A to be the group ring and I_A its augmentation ideal, it is possible to interpret the spaces $I_A^{\otimes_A m}$ as spaces of 'm-singular virtual braids – essentially virtual braids with m 'semi-virtual' double points (subject to a certain equivalence relation) - see [GPV]. Furthermore, the associated quadratic algebra $\mathfrak{pvb}_n$ is a kind of 'infinitesimal' or classical version of PvB_n. One then finds that the map F^0 is a map from singular virtual braids to $\mathfrak{pvb}_n$ which 'forgets topology', in the sense that it forgets all information relating to over- and under-crossings, retaining only the combinatorial information of which strands in the singular virtual braid have semi-virtual double points with which other strands (and the order of the double points). This map F^0 maps global relations (i.e. the R^A_m) to infinitesimal relations (i.e. the $\mathfrak{R}_m$), and global syzygies (*ker* ∂_A) to infinitesimal syzygies (*ker* ∂^{Ind}). However, in general the map F^0 need not be surjective on syzygies. Surjectivity of F^0 on syzygies (the Hutchings Criterion) may be stated informally as:

'Every relation among infinitesimal relations holds also among global relations' (see [Hutchings], section 2.3.2).