Kelmer@Colloq: Spectral gaps for lattices in Lie groups

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Spectral gap for hyperbolic surface.

$\mathbb{H} = \{z = x+iy : y>0\}$ $ds^2 = \frac{dx^2+dy^2}{y^2}$

$SL_2(\mathbb{R}) \circlearrowright \mathbb{H}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$

$\Gamma \subseteq SL_2(\mathbb{R})$ discrete, $X_\Gamma = \Gamma \backslash \mathbb{H}$

Γ is a lattice $\iff$ Γ is discrete, $\mathrm{Vol}(X_\Gamma) < \infty$

$\mathrm{Spec}(\Gamma) = \mathrm{Spec}(X_\Gamma) = \{0 = \underbrace{\lambda_0 < \lambda_1}_{\text{the gap}} \leq \lambda_2 \leq \ldots$

Hyperbolic lattice counting:

Count $N(\Gamma, R) = \#\{\gamma \in \Gamma : \gamma i \in B_R(i)\}$

Thm.

$$N(\Gamma, R) = \frac{|B_R|}{|X_\Gamma|} + \sum_{\lambda_j < \frac{1}{4}} C_j |B_R|^{s_j} + O_\Gamma(|B_R|^{2/3})$$

$\frac{1}{2} < s_j < 1$ $\lambda_j = s_j(1-s_j)$

Aside $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a^2+b^2+c^2+d^2 = 2\cosh(d(\gamma i, i))$

Congruence Groups:

principal cong' gr: $\Gamma(q) = \{\gamma \in SL_2(\mathbb{Z}) : \gamma \equiv I \bmod q\}$

For $\Lambda \subseteq SL_2(\mathbb{Z})$, $\Lambda(q) = \Lambda \cap \Gamma(q)$

Conj (Selberg)

$\lambda_1(\Gamma(q)) \geq \frac{1}{4}$ (Selberg Proved 3/16)

(Kim-Sarnak: 0.238...)

Fact For any $\epsilon > 0$ $\exists \Lambda \subset SL_2(\mathbb{Z})$ w/ $\lambda_1 < \epsilon$

Thm (Xu, Sarnak) For any Λ & q large, $\nexists$ new eigenvalues below 5/36.

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Decay of correlation $\Gamma \subset G = SL_2(\mathbb{R})$, $\varphi \in L^2_0(\Gamma\backslash G)$

$$\Psi_\varphi(g) := \int_{\Gamma\backslash G} \varphi(xg)\overline{\varphi(x)}\,dx \xrightarrow{g\to\infty} 0$$

set

$$p(\Gamma) = \inf\{p \mid \Psi_\varphi \in L^p(G)\ \forall \text{ smooth } \varphi\}$$

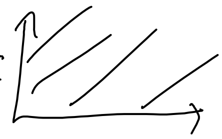
<u>claim</u> $$p(\Gamma) = \begin{cases} \dfrac{2}{1-\sqrt{1-4\lambda_1}} & \lambda_1 < \frac{1}{4} \\ 2 & \lambda_1 \geq \frac{1}{4} \end{cases}$$

More general groups: $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, $\Gamma \subset G$ is irreducible (proj on each side is dense)

Example: $SL_2(\mathbb{Z}[\sqrt{2}]) \subset G$ as

$\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R}\times\mathbb{R}$ via $n+m\sqrt{2} \mapsto (n+m\sqrt{2}, n-m\sqrt{2})$

$X_\Gamma = \Gamma\backslash\mathbb{H}^2$, $\Delta = \Delta_{z_1} + \Delta_{z_2}$

$\text{spec}(\Gamma) = \{(\lambda_1,\lambda_2) : \Delta_{z_i}\varphi = -\lambda_i\varphi\} \subset$ 

$$p(\Gamma) = \frac{2}{1-\sqrt{1-4\tau}} \quad \text{where } \tau = \inf_{\text{spec}-\{0\}} \min(\lambda_1,\lambda_2)$$

Thm. $\psi(\Gamma) > 0$, actually, for many other G's.

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