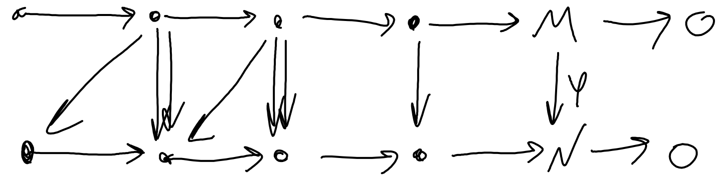
Basic Homological Algebra

December-26-10
7:31 AM



__Claim__ If $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is exact, then so is

$$\underset{}{A \otimes P} \xrightarrow{\iota'} \underset{\mathrm{II}}{B \otimes P} \xrightarrow{\pi'} \underset{\mathrm{I}}{C \otimes P} \to 0$$

__Example.__

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2 \to 0$$

$$0 \not\dashrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0 \qquad \otimes \mathbb{Z}/2 \quad \checkmark$$

$$0 \dashrightarrow \mathbb{Q} \xrightarrow{\cdot 2} \mathbb{Q} \to 0 \to 0 \qquad \otimes \mathbb{Q} \quad \checkmark$$

__Proof__ Exactness at I is obvious.

Exactness at II: Need to show $B \otimes P / \operatorname{im} \iota' \cong C \otimes P$.

$\phi : B \otimes P / \operatorname{im} \iota' \to C \otimes P$ by $[b \otimes p] \mapsto \pi(b) \otimes p$; this is clearly well defined.

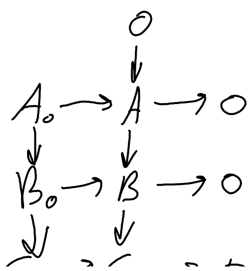
$\psi : C \otimes P \to B \otimes P / \operatorname{im} \iota'$ by $c \otimes p \mapsto [b \otimes p]$, where $\pi(b) = c$.

Well-defined: If $\pi(b') = c$ then $\pi(b - b') = 0$ so $b - b' = \iota a$ so $[b \otimes p] - [b' \otimes p] = [(b - b') \otimes p]$

$$= [\iota'(a \otimes p)] = 0.$$

The long exact sequence.

must be a direct sum of A_0 & B_0 ...



The long exact sequence.

must be a direct sum of A_0 & C_0 ...

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
A_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
B_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
C_0 \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & 0 &
\end{tikzcd}$$