# McCann @ Fields: Optimal Transportation Minicourse: Geometry, Regularity and Applications

October-19-10
2:02 PM

Following http://www.math.toronto.edu/mccann/papers/FiveLectures.pdf

$c(x,y)$ : cost per unit mass transported from $x$ to $y$.

Geodesic space: $\forall x_0, x_1 \ \exists$ path $x_s$ s.t. $d(x_0, x_s) = s\, d(x_0, x_1)$

$$d(x_s, x_1) = (1-s)\, d(x_0, x_1)$$

$\mu^\pm$ measures on $M^\pm$, $\int_{M^\pm} d\mu^\pm = 1$. We're looking

for $G: M_+ \to M_-$ s.t. $G_* \mu_+ = \mu_-$ & s.t.

$$\int c(x, G(x))\, d\mu_+ \text{ is minimal. } =: W_c(\mu^+, \mu^-)$$

$c(x,y)$ could be $|x-y|$, $d(x,y)$, $\underbrace{\tfrac{1}{2}|x-y|^2}_{\text{simpler}}$

<u>Thm</u> If $M_+ = M_- = M$, $c(x,y) = \frac{d^p(x,y)}{p} \Rightarrow$

$0 < p < 1$ $\quad W_c$ is a metric on measures.

$p \geq 1$ $\quad W_c^{1/p}$ is a metric.

<u>Thm</u> For $c = \frac{1}{2}|x-y|^2$, $\exists!$ convex $u: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$

s.t. $G = Du$ $\qquad$ ( $D$: Gradient; $M_\pm$ are $\mathbb{R}^n$ )

PDE. If $\frac{d\mu^\pm}{d\text{vol}} = f^\pm$ then

$$G_* \mu^+ = \mu^- \iff |\det DG(x)| = f^+(x) / f^-(G(x))$$

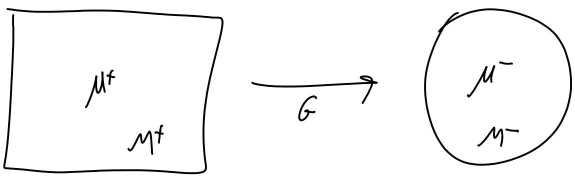
if also $G = Du$, $c = \frac{1}{2}|x-y|^2$ then $\underbrace{\det D^2 u = \frac{f^+(x)}{f^-(Du(x))}}_{\text{Monge-Ampere eqn'.}}$

---

The isoperimetric ineq:

So $\det D^2 u = 1$

So $(\det D^2 u)^{1/n} = 1$



So $\quad 1 \leq \det^{1/n} D^2 u \overset{GM \leq AM}{\leq} \frac{1}{n} \Delta u$

So $\quad vol(M^+) = \int_{M^+} 1 \leq \frac{1}{n} \int_{\partial M} Du(x) \cdot \hat{n}_{M^+}(x) \, dH^{n-1}(x)$

$$\leq \frac{1}{n} H^{n-1}(\partial M^+) \ \ldots$$

---

Kantorovich (1942). Replace $G : M^+ \to M^-$ by

$$\Gamma = \{ 0 \leq \gamma \text{ on } M^+ \times M^- : \pi_{1*}\gamma = \mu^+ , \pi_{2*}\gamma = \mu^- \}$$

$$cost(\gamma) := \int c(x,y) \, d\gamma(x,y)$$

* non-empty
* convex.
* cost is linear.

<u>Def'n</u> $S \subset M^+ \times M^-$ is c-cyclically monotone if $\forall k \in \mathbb{N}$, $\sigma \in S_k$, $(x_1, y_1) \ldots (x_k, y_k) \in S$,

$$\sum c(x_i, y_i) \leq \sum c(x_{\sigma(i)}, y_i)$$

<u>Thm</u> if $\gamma$ is c-optimal in the kantorovich sense then $Supp(\gamma)$ is cyclically monotone. [assuming $c$ is cont.]

<u>Example</u> w/ $c(x,y) = -x \cdot y$ [equiv to $\frac{1}{2}(x-y)^2$]

Cyclic mon $\iff \sum \langle x_i - x_{i-1}, y_i \rangle \geq 0$

<u>Rockefeller's thm</u> $S$ is cyc-mon in above sense iff $\exists$ convex $u$ s.t. $S \subset \{(x, Du|_x)\}$

kantorovich: Duality:

$$\min_{\gamma \in \Gamma} cost(\gamma) = \max_{-(u,v) \in Lip_c} - \int_{M^+} u \, d\mu^+ - \int_{M^-} v \, d\mu^-$$

where $\quad Lip_c = \{ u^\pm : c(x,y) \geq u^+(x) + u^-(y) \}$

Follows from the theory of 2-player 0-sum games:

Follows from the theory of 2-player 0-sum games:

$$\inf_{x \in X} \sup_{y \in Y} p(x,y) \geq \sup_{y \in Y} \inf_{x \in X} \underset{\text{payoff}}{p(x,y)}$$

<u>Thm</u> (von Neuman min-max Thm) If $X, Y$ are convex & $p(x, y_0)$ & $-p(x_0, y)$ are convex, then equality holds in the above.

In the above, take

$$X = \{ 0 \leq \gamma \text{ on } M^+ \times M^- \}$$

$$Y = \{ u^\pm \in L^1(d\mu^\pm) \}$$

$$p(\gamma, (u,v)) = \int_{M^+ \times M^-} (c(x,y) + u(x) + v(y)) \, d\gamma(x,y) - \int_{M^+} u \, d\mu^+ - \int_{M^-} v \, d\mu^-$$

$$= \int c \, d\gamma + \int u (d\gamma - d\mu^+) + \int v (d\gamma - d\mu^-)$$