Chicago ax+b Talk as of September 1

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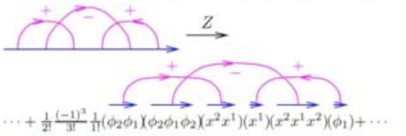
From the $ax+b$ Lie Algebra to the Alexander Polynomial and Beyond

Dror Bar-Natan, Chicago, September 2010 http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

Abstract. I will present the simplest-ever "quantum" formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the "$ax+b$" Lie group). After introducing some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

The 2D Lie Algebra. Let $\mathfrak g = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$, let $\mathfrak g^* = \langle \phi_1, \phi_2\rangle$ with $\phi_i(x^j) = \delta_i^j$, let $I\mathfrak g = \mathfrak g^* \rtimes \mathfrak g$ so $[\phi_i, \phi_j] = [\phi_1, x^1] = 0$ while $[x^1, \phi_2] = -\phi_2$ and $[x^2, \phi_2] = \phi_1$. Let $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak g^* \otimes \mathfrak g \subset$ $I\mathfrak g \otimes I\mathfrak g$. Let $\mathcal U = \{$words in $I\mathfrak g\}/ab - ba = [a,b]$, and let $\hat{\mathcal U}$ be its graded completion with respect to $\deg \phi_i = 1$ and $\deg x^i = 0$. Let $R = \exp(r) \in \hat{\mathcal U} \otimes \hat{\mathcal U}$. ~~Note that $\hat{\mathcal U}/\hat{\mathcal U}\cdot\mathfrak g = \hat{\mathcal S}$ is power series in (ϕ_1, ϕ_2).~~

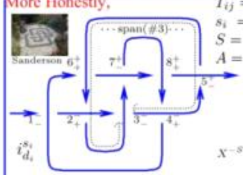
The Invariant. Define Z : {long knots} $\to$ $\mathcal U$ by mapping every $\pm$-crossing to $R^{\pm 1}$; ~~and $\bar Z$ by further projecting mod $\hat{\mathcal U}\cdot\mathfrak g$~~:



$$\cdots + \frac{1}{2!}\frac{(-1)^3}{3!}\frac{1}{1!}(\phi_2\phi_1)(\phi_2\phi_1\phi_2)(x^2x^1)(x^1)(x^2x^1x^2)(\phi_1) + \cdots$$

The Theorem. Z and $\bar Z$ are invariants, and $\bar Z$ is essentially the Alexander polynomial: $\bar Z(K) = N(A(K)(e^{\phi_1}))^{-1}$, where $N = \ldots$ $

More Honestly,



$T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|$,
$s_i = \text{sign}(\#i)$, $d_i = \text{dir}(\#i)$,
$S = \text{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \text{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

Conjecture. For u-knots, A is the Alexander polynomial.

$[x^j, \phi_i] = \delta_i^j \phi_1 - \delta_1^j \phi_i$

$ad(x^j)(\phi_i) = \delta_i^j \phi_1 - \delta_1^j \phi_i$
$= \phi_i(x^j)\phi_1 - \phi_1(x^j)\phi_i$

$ad(x)(\phi) = \phi(x)\phi_1 - \phi_1(x)\phi$

$(ad(x) + \phi_1(x)I)(\phi) = \phi(x)\phi_1$

Prelude: The Euler operator, reduction to a trace problem

The key relations. YII, $T\phi$, $H\phi$, $x\phi$ is central, $x\phi - \phi x = \phi_1$

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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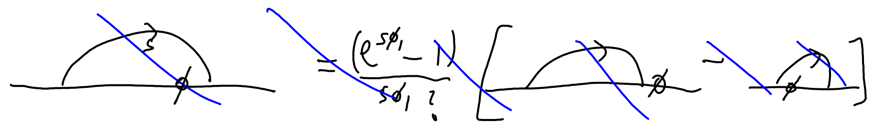
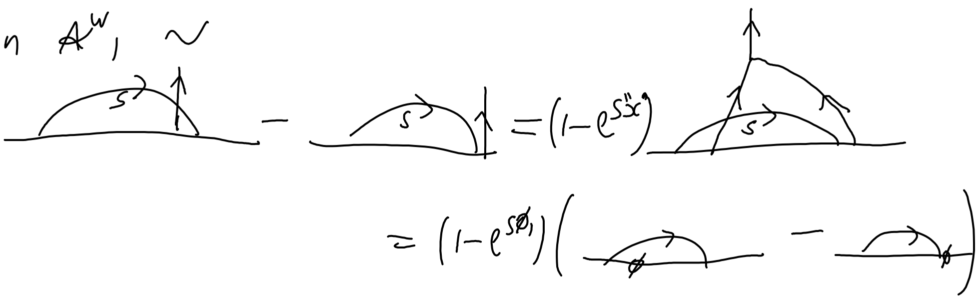
$H\phi$:



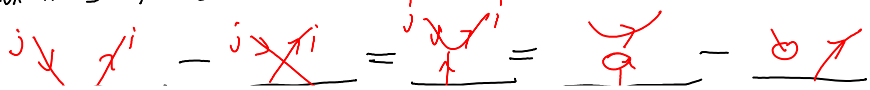
In A^w, ~

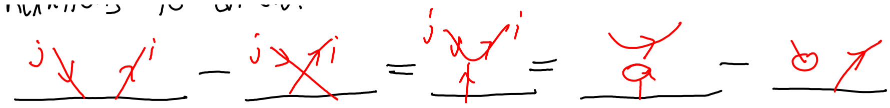


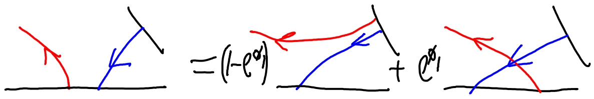
YII:

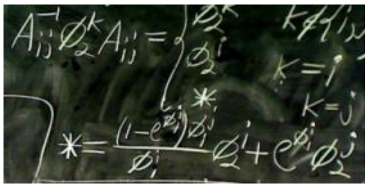
Relations to draw:

$[x^j, \phi_i] = \delta_i^j \phi_1 - \delta_1^j \phi_i$



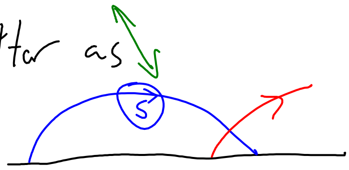


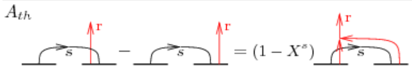




Sep1 Lazyknots BBs.

better as





WKO paper.

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