

Schwartz space: $\int \frac{\partial^k f}{\partial x^k} (1+|x|^k) dx < \infty$

Theorem On Schwartz space, the Fourier transform is determined by the property that it interchanges products with convolutions.

Theorem [Alesker-A-Milman]: Fourier transform is characterized by exchanging product and convolution.
 More precisely: let $F: S \rightarrow S$, bijective, with bijective extension $F': S' \rightarrow S'$ such that
 $F'(f * g) = (Ff) \cdot (F'g)$, $F'(f \cdot g) = (Ff) * (F'g)$
 Then it is the Fourier transform up to a linear term and possibly conjugation.

Here S is
 The Schwartz space
 & S' its dual.

Theorem [A-A-Faifman-M]: For complex valued functions. Assume we are given a bijective map $F: S \rightarrow S$ which admits a bijective extension $F': S' \rightarrow S'$ such that for every $f \in S$ and $g \in S'$ we have
 $F'(f \cdot g) = (Ff) \cdot (F'g)$
 Then there exists a C^∞ -diffeomorphism $u: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that either $Ff = f \circ u$ for all $f \in S$
 or $Ff = \overline{f \circ u}$ $f \in S$

On to the chain rules

$T(f \circ g) = ((Tf) \circ g) Tg$ **A**

Some well known solutions to the equation:

(a) the derivative $Tf = f'$

(b) composition $Tf = \frac{H \circ f}{H}$

(b) product of solutions, powers of solutions.

Theorem B is the only solution of A, in some C^1 setting. In C^0 there are no solution.

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$$Tf = \underbrace{\frac{H \circ f}{H}}_{\text{co-boundary}} \cdot \underbrace{|f'|^p \text{sign}(f')}_{\text{co-homology}} \quad B$$

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