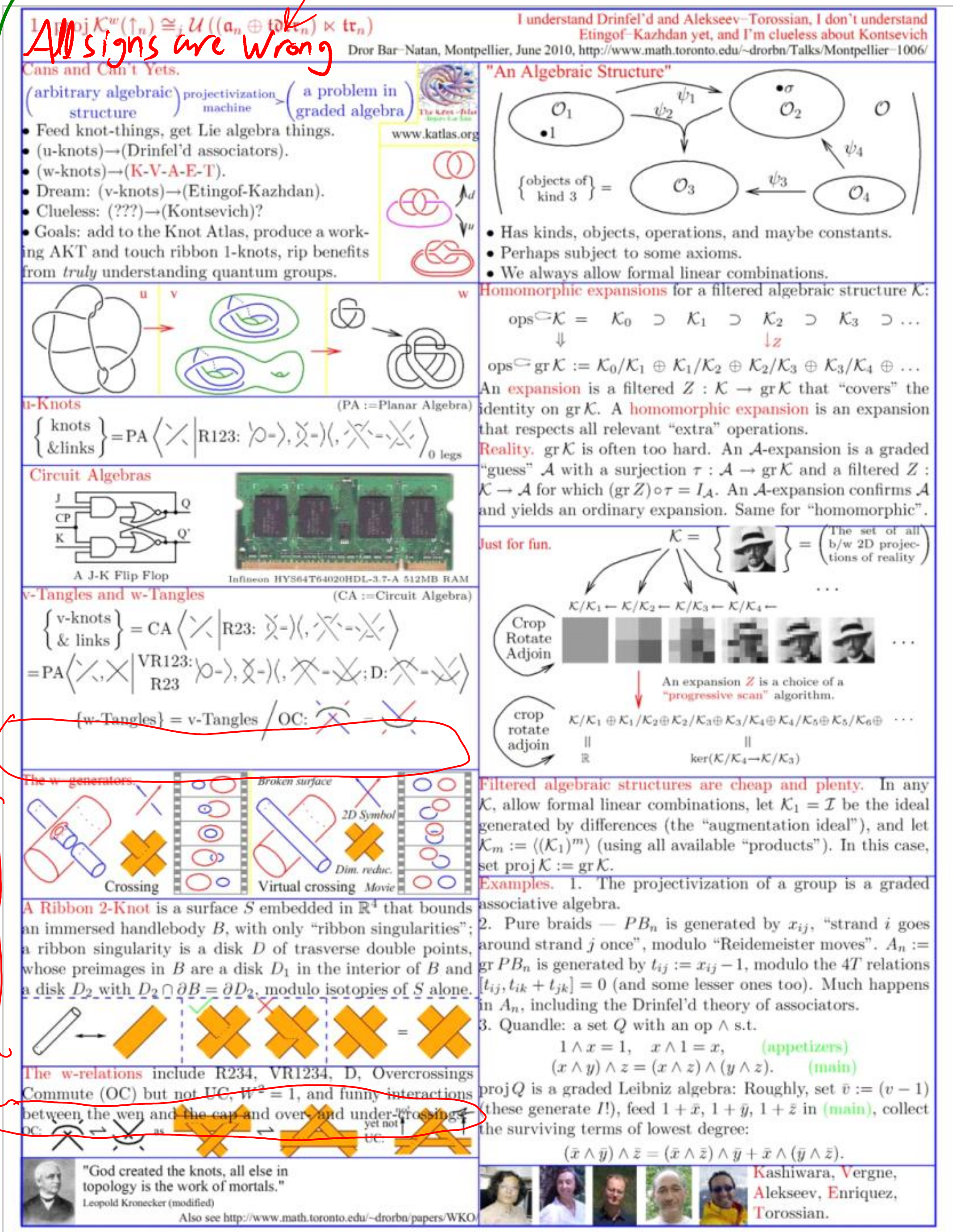
Montpellier Day -5

May-29-10
2:06 PM

in purple, nice font.

All signs are wrong



1. proj $\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$

I understand Drinfel'd and Alekseev–Torossian, I don't understand Etingof–Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar-Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

Cans and Can't Yets.

(arbitrary algebraic structure) $\xrightarrow{\text{projectivization machine}}$ (a problem in graded algebra)

- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel'd associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.

www.katlas.org

u v w

u-Knots (PA := Planar Algebra)

$\left\{\begin{matrix}\text{knots}\\ \&\text{links}\end{matrix}\right\}$ = PA⟨ ╳ | R123: ... ⟩ 0 legs

Circuit Algebras

A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

v-Tangles and w-Tangles (CA := Circuit Algebra)

$\left\{\begin{matrix}\text{v-knots}\\ \&\text{ links}\end{matrix}\right\}$ = CA⟨ ╳ | R23: ... ⟩ = PA⟨ ╳, ╳ | VR123: ..., R23, D: ... ⟩

{w-Tangles} = v-Tangles / OC: ...

The w-generators. Broken surface. 2D Symbol. Dim. reduc. Crossing. Virtual crossing. Movie.

A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:

OC: ... UC: yet not

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

"An Algebraic Structure"

$\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$, $\mathcal{O}$, ψ_1, ψ_2, ψ_3, ψ_4, $\bullet\sigma$, $\bullet 1$; {objects of kind 3} =

- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\text{ops} \circlearrowright \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtered $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Reality. $\operatorname{gr}\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded "guess" $\mathcal{A}$ with a surjection $\tau : \mathcal{A} \to \operatorname{gr}\mathcal{K}$ and a filtered $Z : \mathcal{K} \to \mathcal{A}$ for which $(\operatorname{gr} Z)\circ\tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for "homomorphic".

Just for fun. $\mathcal{K} = \{\ \} =$ (The set of all b/w 2D projections of reality)

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow$

Crop Rotate Adjoin

An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$ ‖ $\mathbb{R}$ ‖ $\ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products"). In this case, set proj $\mathcal{K} := \operatorname{gr}\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.
2. Pure braids — PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.
3. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \quad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad \text{(main)}$$

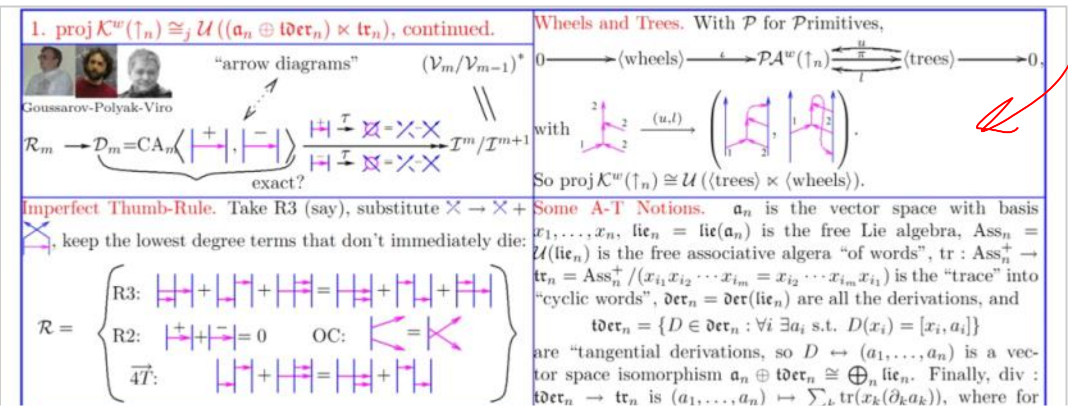
proj Q is a graded Leibniz algebra: Roughly, set $\bar v := (v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), collect the surviving terms of lowest degree:

$$(\bar x \wedge \bar y) \wedge \bar z = (\bar x \wedge \bar z) \wedge \bar y + \bar x \wedge (\bar y \wedge \bar z).$$

Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

http://www.vsaint.com/prince/Prince15.gif

of similar.



1. proj $\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.

"arrow diagrams" $(\mathcal{V}_m/\mathcal{V}_{m-1})^*$

Goussarov-Polyak-Viro

$\mathcal{R}_m \to \mathcal{D}_m = \mathrm{CA}_m\langle \dots \rangle \to \mathcal{I}^m/\mathcal{I}^{m+1}$

exact?

Imperfect Thumb-Rule. Take R3 (say), substitute ╳ → ╳ + ..., keep the lowest degree terms that don't immediately die:

$\mathcal{R} =$ R3: ... ; R2: ... = 0 ; OC: ... ; 4T: ...

Wheels and Trees. With $\mathcal{P}$ for Primitives,

$$0 \to \langle\text{wheels}\rangle \to \mathcal{P}\mathcal{A}^w(\uparrow_n) \rightleftarrows \langle\text{trees}\rangle \to 0$$

with (u,l)

So proj $\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.

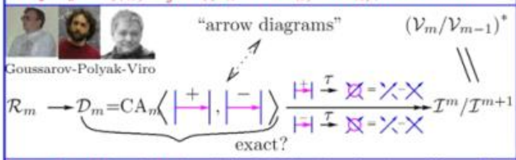
Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \dots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\operatorname{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+/(x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

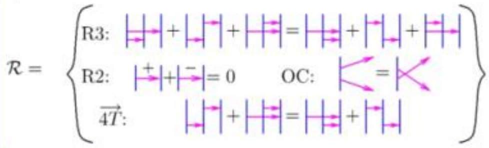
are "tangential derivations, so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, div : $\mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \operatorname{tr}(x_k(\partial_k a_k))$, where for

Similar.

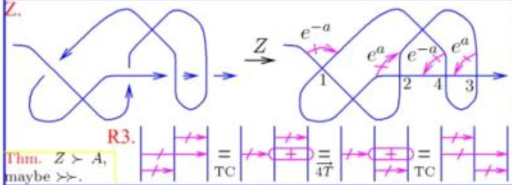
1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.



Imperfect Thumb-Rule. Take R3 (say), substitute ✕ → ✕ + ⋈, keep the lowest degree terms that don't immediately die:

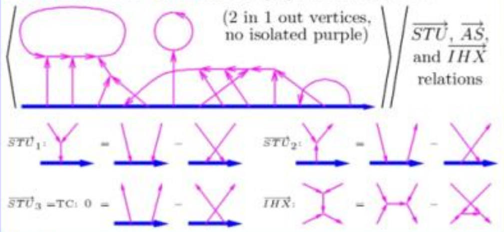


Z.



Thm. $Z \succ A$, maybe $\succ\succ$.

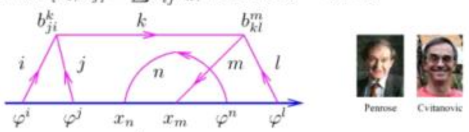
The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to



Proof.

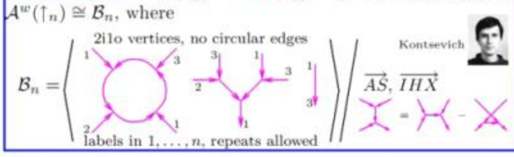
Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

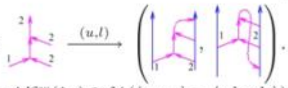
Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where



Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \xrightarrow{\iota} \mathcal{P}\mathcal{A}^w(\uparrow_n) \underset{l}{\overset{u}{\rightleftarrows}} \langle\text{trees}\rangle \longrightarrow 0,$$

with



So $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.

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are "tangential derivations, so $D \leftrightarrow (a_1, \ldots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\operatorname{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \ldots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \operatorname{div} D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\operatorname{div} D = \iota^{-1}(u - l)(D)$, and $e^{uD} e^{-lD} = e^{jD}$.

Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad} x$: $(x\varphi)(y) := \varphi([x, y])$.

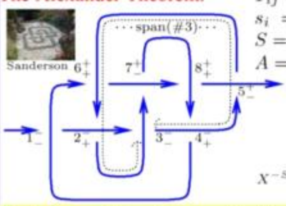
Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\operatorname{div} D$ is $D - D^*$ and $jD = \log(e^D (e^D)^*) = \int_0^1 dt e^{tD} \operatorname{div} D$.

Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.

Theorem. $\mathfrak{sder}_n = \pi\alpha(\operatorname{proj}\text{u-tangles})$, where α is the obvious map proj u-tangles → proj w-tangles.

Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$, $s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$, $S = \mathrm{diag}(s_i d_i)$, $A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \mathrm{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[[x]]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan

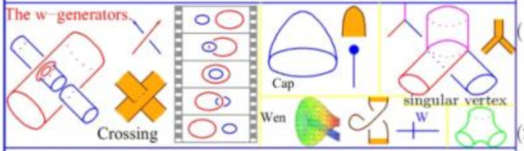
I understand Drinfel'd and Alekseev–Torossian, I don't understand Etingof–Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar–Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

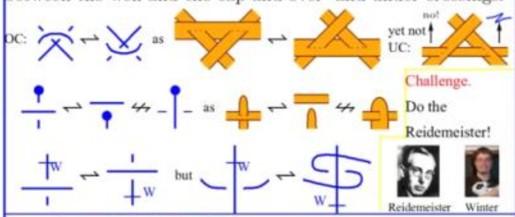
Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \text{w-generators} \,\middle|\, \text{w-relations} \,\middle|\, \text{unary w-operations} \right\rangle$$

The w–generators.

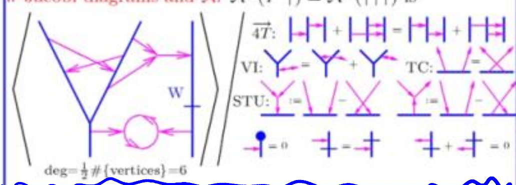


The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:

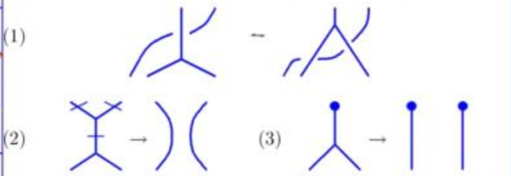


The unary w–operations

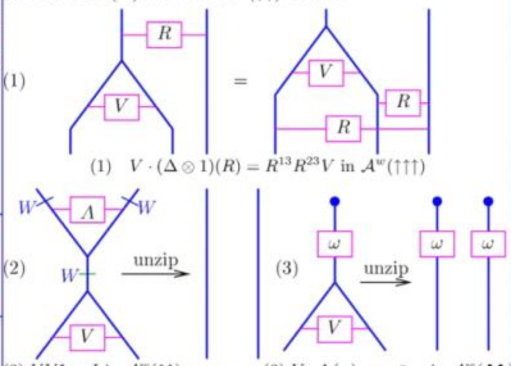
w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \overset{\leftarrow}{\vdash\!\dashv} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V \cdot (\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$

(3) $V \cdot \Delta(\omega) = \omega \otimes \omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F \in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad \text{and} \quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$,
(2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c} e^{jD}$, and
(3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$
$\Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

Thick blue

Claim Associators $\leftrightarrow$ Sol'ns of KV.

An even bigger algebraic structure

(green knotted trivalent graphs.) $\longrightarrow$ (knotted tubes & strings in $\mathbb{R}^4$)

2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan, continued.

or "we ignore the hexagon".

Texify, add hexagon

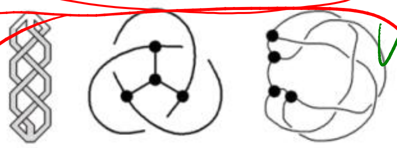
$$(A\otimes B)\otimes C \xrightarrow[\text{in } \mathcal{U}(\mathfrak{g})^{\otimes 3}]{\Phi} A\otimes(B\otimes C)$$

$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[rr, "(\Delta 1 1)\Phi"] \arrow[d, "\Phi\otimes 1"'] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "(1 1 \Delta)\Phi"] \\
(A\otimes(B\otimes C))\otimes D \arrow[dr, "(1\Delta 1)\Phi"'] & & A\otimes(B\otimes(C\otimes D)) \\
& A\otimes((B\otimes C)\otimes D) \arrow[ur, "1\otimes\Phi"'] &
\end{tikzcd}$$

Φ is an "associator"

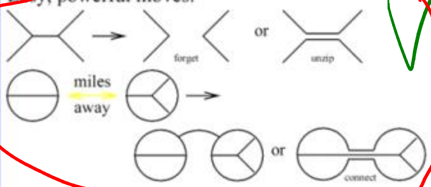
$$(\Phi\otimes 1)\cdot(1\Delta 1)\Phi\cdot(1\otimes\Phi) = (\Delta 1 1)\Phi\cdot(1 1\Delta)\Phi$$

"The Drinfel'd Pentagon"

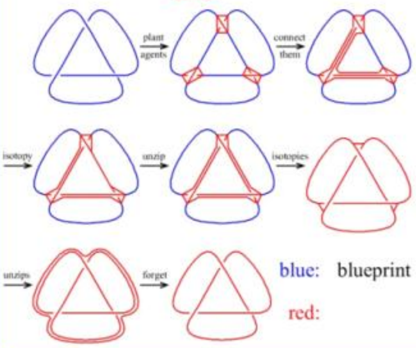


~~Need a new relation:~~

Easy, powerful moves:



Using moves, KTG is generated by ribbon twists and the tetrahedron  :



blue: blueprint

red:

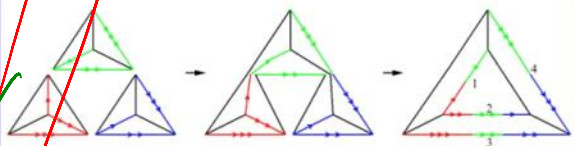
Modulo the relation(s): 

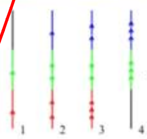
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

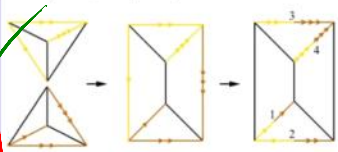
add headers & separators.

Proof.

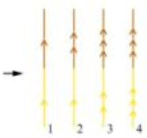
 $:= \Phi \in \mathcal{A}(\uparrow_3)$



 $= (\Phi\otimes 1)\cdot(1\otimes\Delta\otimes 1)(\Phi)\cdot(1\otimes\Phi) \in \mathcal{A}(\uparrow_4)$



Drinfeld

 $= (\Delta\otimes 1\otimes 1)(\Phi)\cdot(1\otimes 1\otimes\Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Free the right column!

* See that A-B pair appear across folds. ✓