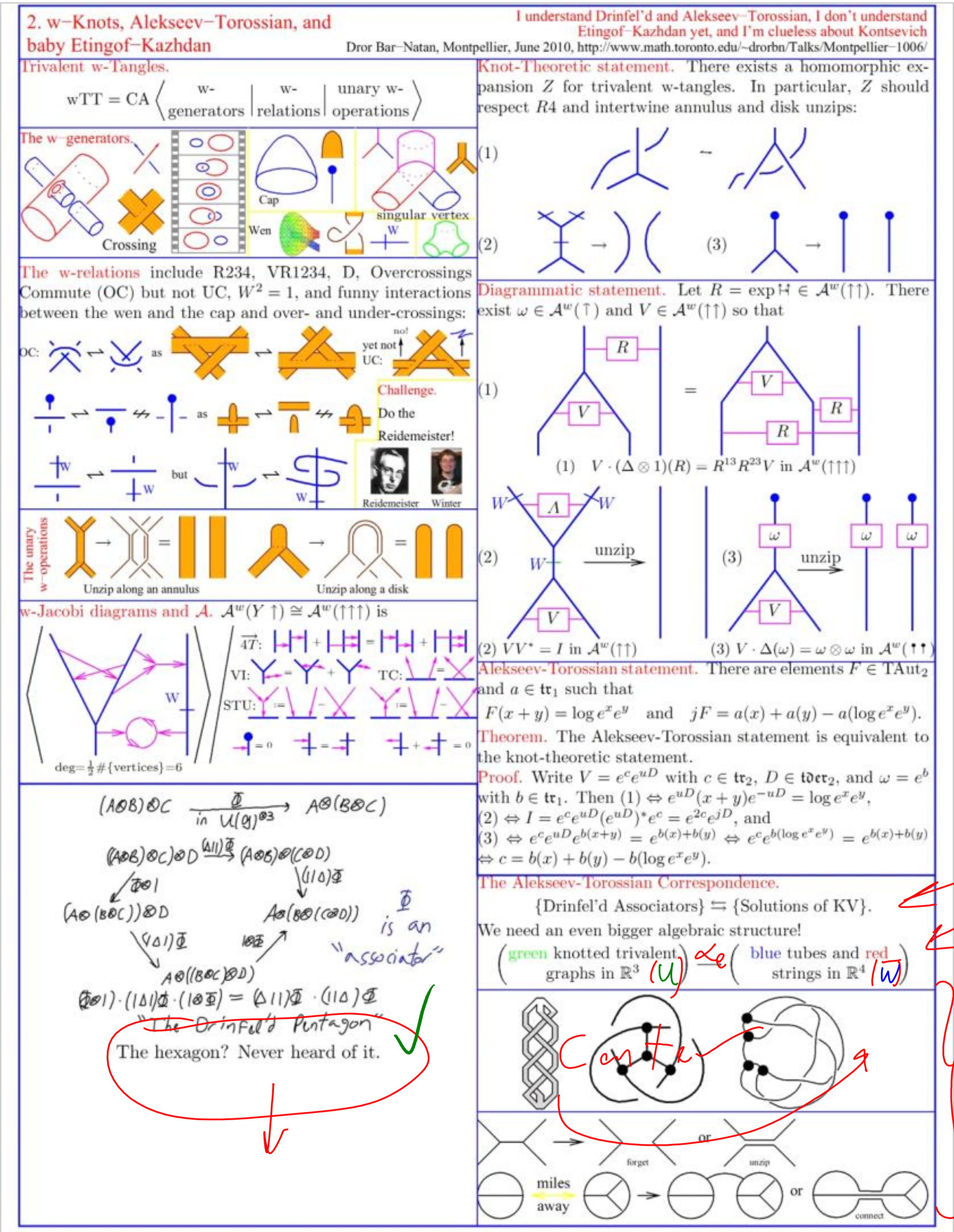
Montpellier Day -4

May-30-10
11:24 AM



2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan

I understand Drinfel'd and Alekseev–Torossian, I don't understand Etingof–Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar-Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/

Trivalent w-Tangles. wTT = CA ⟨ w-generators | w-relations | unary w-operations ⟩

The w-generators. Crossing, Cap, Wen, singular vertex

The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2=1$, and funny interactions between the wen and the cap and over- and under-crossings:

OC: … as … yet not UC: … no!

… but …

Challenge. Do the Reidemeister!

Reidemeister Winter

The unary w-operations

Unzip along an annulus Unzip along a disk

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is

$\overrightarrow{4T}$: … VI: … TC: … STU: …

$\deg=\frac12\#\{\text{vertices}\}=6$

$(A\otimes B)\otimes C \xrightarrow[\text{in } U(\mathfrak g)^{\otimes 3}]{\Phi} A\otimes(B\otimes C)$

$((A\otimes B)\otimes C)\otimes D \xrightarrow{(\Delta 11)\Phi} (A\otimes B)\otimes(C\otimes D) \xrightarrow{(11\Delta)\Phi} A\otimes(B\otimes(C\otimes D))$

$((A\otimes B)\otimes C)\otimes D \xrightarrow{\Phi\otimes 1} (A\otimes(B\otimes C))\otimes D \xrightarrow{(1\Delta 1)\Phi} A\otimes((B\otimes C)\otimes D) \xrightarrow{1\otimes\Phi} A\otimes(B\otimes(C\otimes D))$

Φ is an "associator"

$(\Phi\otimes 1)\cdot(1\Delta 1)\Phi\cdot(1\otimes\Phi) = (\Delta 11)\Phi\cdot(11\Delta)\Phi$

"The Drinfel'd Pentagon"

The hexagon? Never heard of it.

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect R4 and intertwine annulus and disk unzips:

(1) (2) → (3) →

Diagrammatic statement. Let $R=\exp\,\overset{\to}{\vdash\!\dashv}\in\mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega\in\mathcal{A}^w(\uparrow)$ and $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that

(1) $V\cdot(\Delta\otimes 1)(R)=R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) unzip $VV^*=I$ in $\mathcal{A}^w(\uparrow\uparrow)$

(3) unzip $V\cdot\Delta(\omega)=\omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F\in\mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

$$F(x+y)=\log e^xe^y \quad\text{and}\quad jF=a(x)+a(y)-a(\log e^xe^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

Proof. Write $V=e^ce^{uD}$ with $c\in\mathfrak{tr}_2$, $D\in\mathfrak{tder}_2$, and $\omega=e^b$ with $b\in\mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD}=\log e^xe^y$,
(2) $\Leftrightarrow I=e^ce^{uD}(e^{uD})^*e^c=e^{2c}e^{jD}$, and
(3) $\Leftrightarrow e^ce^{uD}e^{b(x+y)}=e^{b(x)+b(y)} \Leftrightarrow e^ce^{b(\log e^xe^y)}=e^{b(x)+b(y)}$
$\Leftrightarrow c=b(x)+b(y)-b(\log e^xe^y)$.

The Alekseev-Torossian Correspondence.

{Drinfel'd Associators} ⇆ {Solutions of KV}.

We need an even bigger algebraic structure!

(green knotted trivalent graphs in $\mathbb{R}^3$ (u)) $\hookrightarrow$ (blue tubes and red strings in $\mathbb{R}^4$ (w))

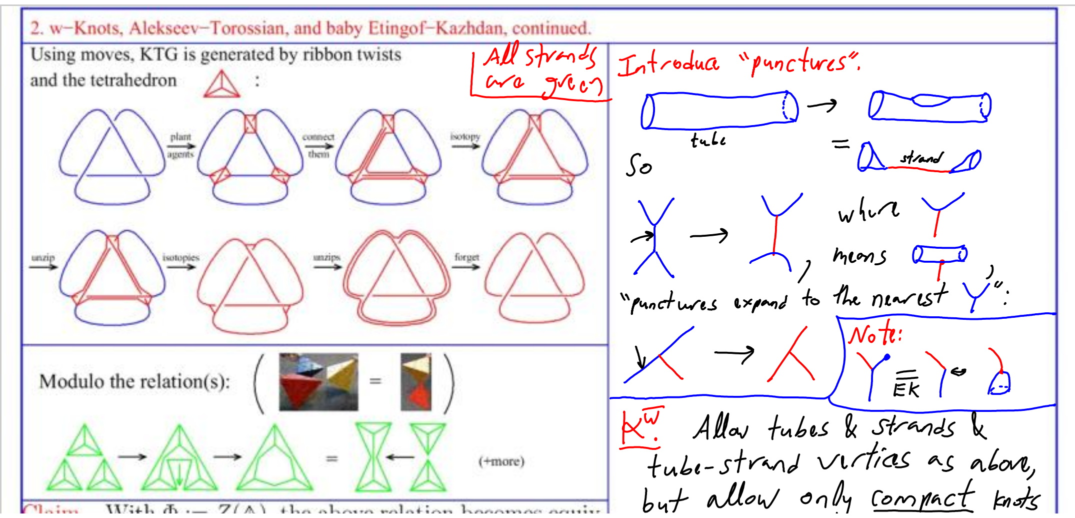
Center

miles away → forget or unzip … or connect

name

KTW

Turn green



2. w–Knots, Alekseev–Torossian, and baby Etingof–Kazhdan, continued.

Using moves, KTG is generated by ribbon twists and the tetrahedron :

plant agents → connect them → isotopy → unzip → isotopies → unzips → forget →

Modulo the relation(s): (=)

(+more)

Claim. With $\Phi:=Z(\triangle)$ the above relation becomes equiv…

All strands are green

Introduce "punctures".

tube → = strand

So → , where means ,

"punctures expand to the nearest Y":

→

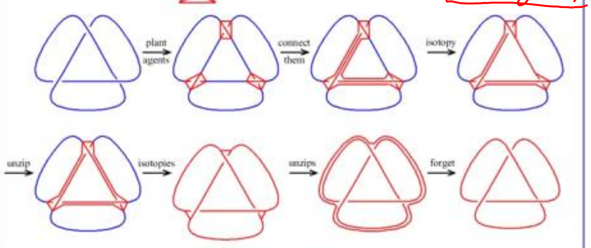
Note: $\overset{=}{EK}$

K^w: Allow tubes & strands & tube-strand vertices as above, but allow only compact knots

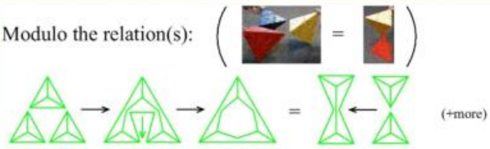
2✓

Using moves, KTG is generated by ribbon twists and the tetrahedron :

All strands are green

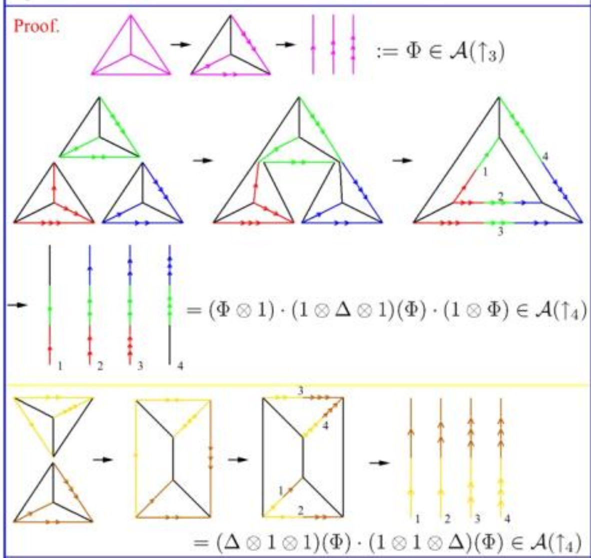


Modulo the relation(s):



Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof.

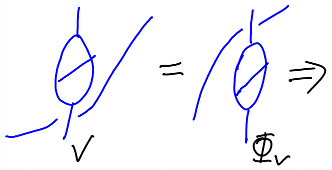


$$:= \Phi \in \mathcal{A}(\uparrow_3)$$

$$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$$

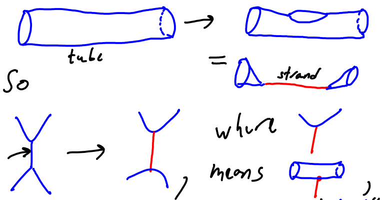
$$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$$

{SolKV} → {Associators}: Trivial – a tetrahedron has 4 vertices.

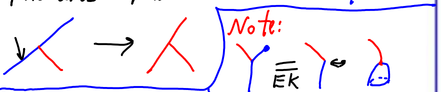


$\Phi_v \in \mathfrak{sder}_3$, so it is a sum of unrooted trees.

Introduce "punctures".



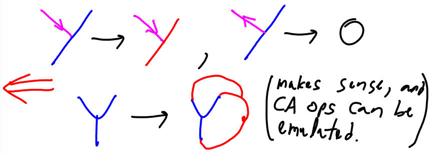
"punctures expand to the nearest Y":



$K^{\bar w}$: Allow tubes & strands & tube-strand vertices as above, but allow only compact knots – nothing runs to ∞.

$K^w \Leftrightarrow K^{\bar w}$, Claim K^w has a homomorphic expansion iff $K^{\bar w}$ has a homomorphic expansion.

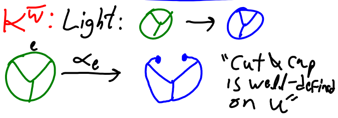
⟹ Puncture $\mathcal{A}$ & Z:



⟸ (makes sense, and CA ops can be emulated.)

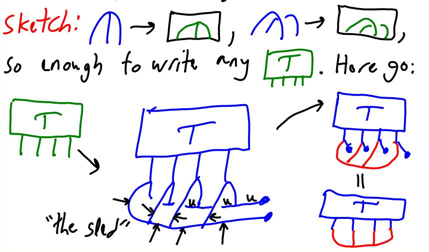
$K^u \to K^{\bar w}$: Light:

Better: "Cut & cap is well-defined on u"



Theorem. The generators of $K^{\bar w}$ can be written in terms of the generators of K^u (I.e., given Φ, can write a formula for V).

Sketch: so enough to write any [T]. Here go:



$$\begin{array}{ccccc} (u\text{-KTGs}) & \xrightarrow{\alpha} & (w\text{-TT}) & \xrightarrow{Z} & \mathcal{A}^w \\ \downarrow & & \downarrow & & \\ (u\text{-TT}) & & (\bar w\text{-TT}) & \xrightarrow{Z} & \mathcal{A}^{\bar w} \end{array}$$