Schick's Problem on RE sets

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For a set $A \subset \mathbb{N}$, let $i_A : \mathbb{N} \to \mathbb{N}$ be
$i_A(n) = |A \cap [1, n]|$ (so for $n \in A$, $i_A(n)$ is the serial number of n in A)

and let $f_A(n) := n + i_A(n)$.

Question Assume A is an R.E. set. Is $B = f_A(A)$ also R.E.?

i.e., to each element of A add its index within A.

Strategy A B is not RE. Assume it is and use it to show that A is recursive.

Let α be a machine that outputs A and assume β is a machine that outputs B.

Goal! Certify with a certainty of stopping [or failing] that some output n of α, or $n+1$ of β, is the smallest that will ever be produced by its respective machine.

notation: $\alpha \to n$

— If $\beta \to 2$, done

If $\beta \to 3$, wait for $\alpha \to 1$ or $\alpha \to 2$; one of the two must happen and in either case we are done:

$\alpha \to 1 \Rightarrow 2 \in B \Rightarrow \beta \to 3$ isn't minimal

$\alpha \to 2 \Rightarrow \begin{cases} \alpha \to 1 \Rightarrow 2, 4 \in B, 3 \notin B, \Rightarrow\Leftarrow \\ \text{So } \alpha \to 2 \ \& \ \beta \to 3 \text{ are minimal.} \end{cases}$

If $\beta \to 4$, wait for $\alpha \to 1, 2,$ or 3.

$\alpha \to 1$ or $2 \Rightarrow \beta \to 4$ isn't minimal.

$\alpha \to 3 \Rightarrow A \cap [1,3] = \{3\}$ or $\cancel{\{1,3\}}$ or $\cancel{\{2,3\}}$ or $\{1,2,3\}$

$B = 4, 6+$

$B = 2, 4, 6, 8+$

$\beta \to 2$

$\beta \not\to 7$

Goal 2 Will $\alpha \to 1$?

Strategy B B is RE.

Itai's Solution:

Question: For a subset B of N (the naturals), write B={b_1<b_2<b_3<...} and let TB={k+b_k}. If B is RE, is it always true that TB is RE?

Answer: No. Let A be the set of n's such that Turing machine number n halts, and let B={2^n: n\in A}. Then B is clearly RE, but TB is not. Indeed, elements of TB are of the form 2^n+k, where 0<=k<=n, so k is much smaller than 2^n. Thus if you get an element of TB you can immediately find its n and k. But k is the number of machines before n that stop. If you know that number it is easy to find out which are the machines that stop - you simply run all n machines in parallel until exactly k of them stop, and you now know that the rest will never stop. So if you have a machine that can produce arbitrarily large elements of TB then you can solve the halting problem; so TB is not RE.