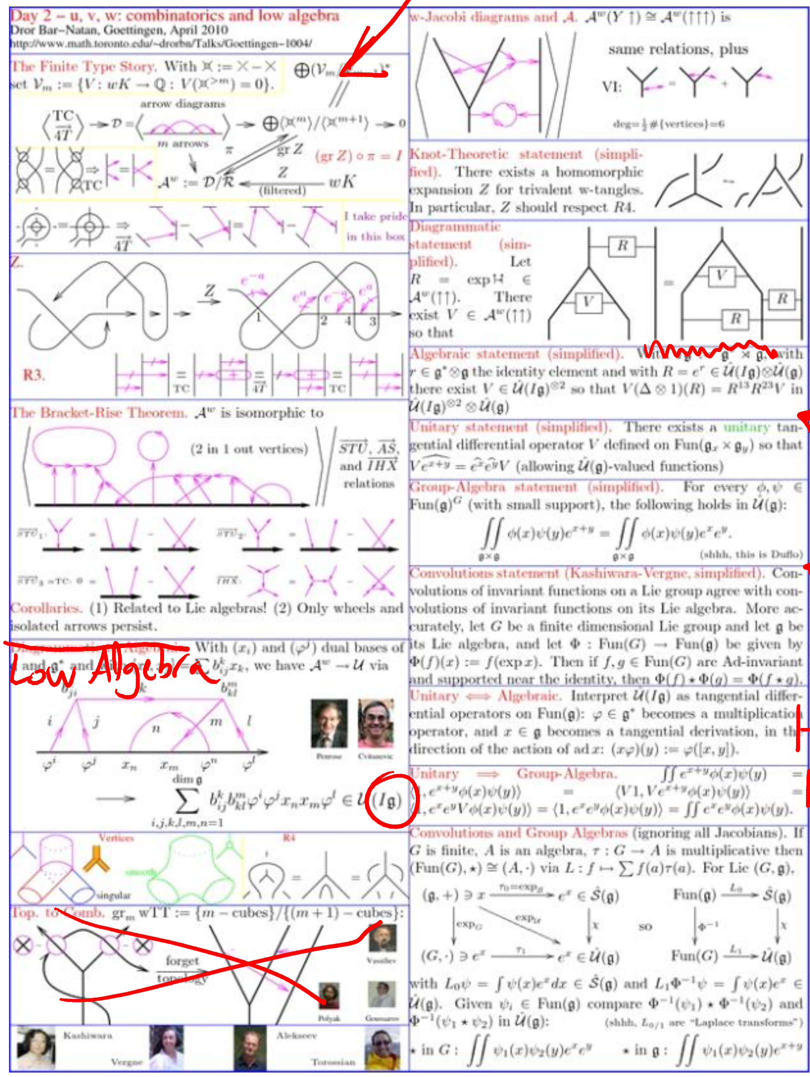
Goettingen - Tuesday preps

April-27-10
1:08 PM



Day 2 – u, v, w: combinatorics and low algebra
Dror Bar-Natan, Goettingen, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Goettingen-1004/

The Finite Type Story. With $\asymp := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\asymp^{>m}) = 0\}$. $\bigoplus (\mathcal{V}_m/\mathcal{V}_{m-1})^*$

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is same relations, plus VI:

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.

Diagrammatic statement (simplified). Let $R = \exp \dashv \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that

Algebraic statement (simplified). With $r \in \mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g})\otimes\hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to (2 in 1 out vertices) / $\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Group-Algebra statement (simplified). For every $\phi,\psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$

(shhh, this is Duflo)

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f)\star\Phi(g) = \Phi(f\star g)$.

Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Low Algebra

With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i,x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$: $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator, and $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.

Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y\phi(x)\psi(y)\rangle = \iint e^x e^y\phi(x)\psi(y)$.

$I\mathfrak{g} := \mathfrak{g}^*\rtimes\mathfrak{g}$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G),\star) \cong (A,\cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G,\mathfrak{g})$,

with $L_0\psi = \int\psi(x)e^x dx \in \hat{S}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int\psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

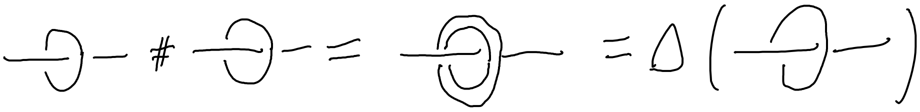
$\star$ in G: $\iint\psi_1(x)\psi_2(y)e^x e^y$ $\quad\star$ in $\mathfrak{g}$: $\iint\psi_1(x)\psi_2(y)e^{x+y}$

Top. to Comb. $\mathrm{gr}_m\,wTT := \{m-\text{cubes}\}/\{(m+1)-\text{cubes}\}$: forget topology

Kashiwara, Vergne, Alekseev, Torossian, Petrov, Cvitanovic, Vasiliev, Polyak, Goussarov

1. K (top) $\xrightarrow{Z:\text{high}}$ $A=$ proj K (comb.) $\xrightarrow{g_1} U(g_1)$, $\xrightarrow{g_2} U(g_2)$, $\xrightarrow{g_3} U(g_3)$ — Low

2. $1+1=2$ on an abacus



$\xrightarrow[\text{u-machine}]{\text{the}}$ The Duflo isomorphism $S(g)^g \simeq U(g)^g$