

Goettingen - Sunday preps

April-25-10
7:15 AM

<p>Day 1 - u, v, w: topology and philosophy Dror Bar-Natan, Goettingen, April 2010</p>	<p>u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra http://www.math.toronto.edu/~drobn/Talks/Goettingen-1004/</p>
<p>Plans and Dreams (arbitrary algebraic structure) $\xrightarrow{\text{projectivization machine}}$ (a problem in graded algebra)</p> <ul style="list-style-type: none"> • Feed knot-things, get Lie algebra things. • Feed u-knots, get Drinfel'd associators. • Feed w-knots, get Kashiwara-Vergne-Alekseev-Torossian. • Dream: Feed v-knots, get Etingof-Kazhdan. • Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology. 	
<p>u-Knots (PA := Planar Algebra) $\{\text{knots} \& \text{links}\} = \text{PA} \langle \text{R123} : \text{O}, \text{X} \rangle_{0 \text{ legs}}$</p>	<p>Circuit Algebras A J-K Flip Flop</p>
<p>v-Knots (CA := Circuit Algebra) $\{\text{v-knots} \& \text{links}\} = \text{CA} \langle \text{R23} : \text{X}, \text{Y} \rangle_{0 \text{ legs}}$</p>	<p>"An Algebraic Structure"</p>
<p>w-Tangles $\{\text{w-Tangles}\} = \text{v-Tangles} / \text{OC}$</p>	<p>• Has kinds, objects, operations, and maybe constants. • Perhaps subject to some axioms. • We always allow formal linear combinations.</p>
<p>The w-generators Broken surface, 2D Symbol, Dim. reduc, Crossing, Virtual crossing, Cap, Wen, Vertices</p>	<p>Homomorphic expansions for a filtered algebraic structure \mathcal{K}:</p> <p>$\text{ops} \subseteq \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$</p> <p>$\text{ops} \subseteq \text{gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$</p>
<p>A Ribbon 2-Knot is a surface embedded in \mathbb{R}^4 that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of transverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.</p>	<p>An expansion is a filtration respecting $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$ that "covers" the identity on $\text{gr } \mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.</p>
<p>The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:</p>	<p>Just for fun.</p> <p>$\mathcal{K} = \left\{ \begin{array}{c} \text{The set of all} \\ \text{h/w 2D} \\ \text{projections} \\ \text{of reality} \end{array} \right\}$</p> <p>Crop Rotate Adjoin</p> <p>crop rotate adjoin</p> <p>An expansion Z is a choice of a "progressive scan" algorithm.</p>
<p>Challenge: Do the register</p>	<p>Filtered algebraic structures are cheap and plenty. In any \mathcal{K}, allow formal linear combinations, let \mathcal{K}_1 be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").</p>
<p>Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op \wedge s.t.</p> <p>$1 \wedge x = 1, \quad x \wedge 1 = x, \quad \text{(appetizers)}$ $(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z), \quad \text{(main)}$</p>	<p>proj Q is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I), feed $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:</p> <p>$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$</p>
<p>1) http://qlink.queensu.ca/~4b11/interesting.html Also see http://www.math.toronto.edu/~drobn/papers/WKO/</p>	<p>add: \sim preview of day 2.</p> <p>* Add a \checkmark</p> <p>$\mathcal{K} \xrightarrow{\text{high}} \mathcal{K} \xrightarrow{\text{low}} \text{Lie}$</p> <p>box.</p>

Day 2 - u, v, w: combinatorics and low algebra
Dror Bar-Natan, Goettingen, April 2010
<http://www.math.ksu.edu/~dobar/Talks/Goettingen-1004>

u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra

4.5: From dg 1

Top. to Comb. $gr_m wTT := (m - cubes) / ((m+1) - cubes)$

forget topology

w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y) \cong \mathcal{A}^w(\{\{\}\})$ is

Diagrammatic to Algebraic. With (x_i) and (z^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $\{x_i, x_j\} = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via

The Finite Type Story. With $\mathcal{K} := \times - \times \oplus (V_m/V_{m-1})^*$ set $V_m := \{V : wK \rightarrow Q : V(\mathbb{R}^{2m}) = 0\}$.

arrow diagrams

on arrows

on the prisms in this box

R3.

The Bracket-Rise Theorem. \mathcal{A}^w is isomorphic to

(2 in 1 out vertices) STU, AS, and IHX relations

Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

- Finite type invariants, weight systems, chord diagrams, arrow diagrams, 4T relations
- The "bracket-rise" theorem, STU and IHX relations
- Maps into various kinds of universal enveloping algebras.

*all arrows go purple.

Day 3 - u, v, w: high algebra
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u, v, and w-Knots: Topology, Combinatorics and Low and High Algebra

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect R4 and intertwine annulus and disk unknots:

(1)

(2)

Diagrammatic statement. Let $R = \exp^{\text{Lie}} \in \mathcal{A}^w(\{\{\}\})$. There exist $\omega \in \mathcal{A}^w(\{\{\}\})$ and $V \in \mathcal{A}^w(\{\{\}\})$ so that

(1)

(2)

Algebraic statement. With $\mathfrak{J}\mathfrak{g} := \mathfrak{g}^* \otimes \mathfrak{g}$, with $\epsilon : \mathcal{U}(\mathfrak{J}\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\mathcal{U}(\mathfrak{g})$, with W the automorphism of $\mathcal{U}(\mathfrak{J}\mathfrak{g})$ induced by flipping the sign of \mathfrak{g}^* , with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = \epsilon^r \in \mathcal{U}(\mathfrak{J}\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ there exist $\omega \in \mathcal{S}(\mathfrak{g}^*)$ and $V \in \mathcal{U}(\mathfrak{J}\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{\otimes 2} P^{\otimes 2} V$ in $\mathcal{U}(\mathfrak{J}\mathfrak{g})^{\otimes 2} \otimes \mathcal{U}(\mathfrak{g})$

(2) $V \cdot \text{SBIV} = 1$ (3) $r \otimes \epsilon(V \Delta \omega) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^{\otimes 2}$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_* \times \mathfrak{g}_*)$ so that

(1) $V \epsilon^{\otimes 2} \omega = \epsilon^{\otimes 2} \omega V$ (allowing $\mathcal{U}(\mathfrak{g})$ -valued functions)

(2) $V V^* = I$ (3) $V_{\omega \otimes \omega} = \omega \otimes \omega$

Group-Algebra statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^{\otimes 2}$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^{\otimes 2}$ (with small support), the following holds in $\mathcal{U}(\mathfrak{g})$:

$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y) \omega_{x,y}^{\otimes 2} \epsilon^{\otimes 2} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y) \omega_{x,y}^{\otimes 2} \epsilon^{\otimes 2}.$$

(obdd, $\omega^* = j^{\otimes 2}$) (obdd, this is Dufo)

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x) f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) * \Phi(g) = \Phi(f * g).$$

- Kashiwara-Vergne and Alekseev-Torossian: convolutions, integrals, measure preserving transformations, unitary operators, universal formulas and universal equations.
- A word on knotted trivalent graphs, Drinfel'd associators, and Chern-Simons-Witten theory.
- Dreams on v-knots, Etingof-Kazhdan, and quantization of Lie bi-algebras.
- Hallucinations on knot homologies and on further physics.

* Bring material from Day 1

ouch.